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The GVAR Approach and the Dominance of the US Economy*

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Abstract

This paper extends the recent literature about global macroeconomic modelling by allowing the presence of a globally dominant economy. Our contribution is both theoretical and empirical. From a theoretical standpoint, we follow Chudik and Pesaran (2011 and 2012) to derive the GVAR approach as an approximation to two Infinite-Dimensional VAR (IVAR) models featuring nonstationary variables: one corresponding to the world consisting of several small open economies (benchmark model), and one corresponding to the world featuring a dominant economy (extended model). It is established that in the presence of a dominant economy, restrictions implied by the asymptotic analysis of a system without a dominant economy are no longer valid. The theoretical framework is then brought to the data by estimating both versions of the GVAR model featuring 33 countries for the period 1979(Q2)-2003(Q4). We found some support for the extended version of the GVAR model, allowing the US to be the dominant player in the world economy.

JEL codes: C32, E17, F47

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1 Introduction

Macroeconomic modelling in a globalized world is a challenging area of research. Individual economies are interdependent with one another and thus all macroeconomic variables in the global economy are jointly determined. The time span of available data, however, is nowhere near large enough to consider scores of advanced economies, each featuring several key macroeconomic indicators, in a single unrestricted VAR due to 'curse of dimensionality'.

One possible method of controlling the proliferation of parameters is found within the context of Bayesian estimation where an unrestricted VAR is combined with Bayesian priors (Canova and Ciccarelli, 2007, among others). A more common practice in the literature to overcome the curse of dimensionality is to impose restrictions directly on the parameters of the model. In particular, all foreign economies are typically approximated by one representative economy constructed as a (trade-)weighted average of foreign economies. Such 'rest of the world' aggregate variables are generally referred to as 'star' variables and typically treated as exogenous to the home economy. This modelling approach is labelled in the literature as 'small open economy' framework (see Schmitt-Grohe and Uribe, 2003). Examples include macroeconometric modelling of a single open economy (for instance Adolfson et al., 2007, provide an application for the Euro area), equilibrium real exchange rate modeling (e.g. Bussière et al., 2010), or the recently introduced coherent framework for modelling international linkages known as GVAR, originally developed by Pesaran, Schuermann, and Weiner (PSW 2004, with discussion).[1] PSW estimate individual country VARX* models, each featuring domestic endogenous variables and weakly exogenous foreign star variables, and then combine them in a single global VAR (GVAR) featuring all variables in the panel.

Until recently, little justification has been given in the literature for the above mentioned conventional restrictions that empirical researchers typically impose in applied macroeconomic modelling, besides referring to the home economy as small and open. Chudik and Straub (2011) demonstrate within the context of an N-country Dynamic Stochastic General Equilibrium (DSGE) model that such a modelling strategy is asymptotically (as N becomes large) justified only if no country is locally or globally dominant.[2] This paper departs from existing empirical literature by allowing for the presence of a globally dominant economy.[3]

Following Chudik and Pesaran (2011 and 2012), we control the proliferation of parameters by imposing restrictions that are binding in the limit as $N \to \infty$ and we derive the GVAR model as an approximation to an infinite-dimensional VAR (IVAR) where all variables in the global economy

[1] The GVAR approach is capable of capturing cointegration within as well as across countries. It has been used to analyse credit risk in Pesaran et al. (2006) and Pesaran, Schuermann, and Treutler (2006). An extended and updated version of the GVAR by Dées et al. (2007), which treats the Euro area as a single unit, was used by Pesaran, Smith, and Smith (2007) to evaluate the UK and the Sweden entry into the Euro. Further developments of a global modelling approach are provided in Pesaran and Smith (2006). Garratt et al. (2006) provide a textbook treatment of the GVAR.

[2] The concept of local and global dominance within the context of an N-country DSGE model is defined according to the orders of magnitude of the individual elements of export and import share matrices. See Chudik and Straub (2011) for details. The international trade flows in theoretical models depend on various factors such as the size and position of individual economies in the world, preferences, degree of international specialization *etc.*

[3] Shocks originating in US are viewed possibly as a global shocks in Dées et al. (2007) version of GVAR, where the dominant role of US economy is confined to endogenous determination of oil prices in the US model and to exclusion of US-specific star variables that did not pass weak exogeneity tests from the US VARX* model.

are integrated of order one and determined endogenously. Two IVAR models are considered. The first corresponds to a world consisting of several small, open economies. Macroeconomic variables in this model are cross sectionally weakly dependent once conditioned on unobserved and observed common factors such as technological progress, oil prices, etc.[4] An idiosyncratic shock to any economy is not a common factor. This means that strong cross section dependence, as defined by Chudik, Pesaran, and Tosetti (2011), is not due to the influence of a particular country. The second IVAR model corresponds to a world consisting of one dominant economy and several small open economies. This paper shows that in the presence of a dominant economy, data shrinkage (restrictions) implied by the asymptotic analysis of a system without a dominant economy are no longer valid. In particular, individual country models need to be augmented by variables of the dominant economy. This suggestion is explored in the empirical section of the paper.

Two versions of the GVAR model featuring real output, short term interest rates, inflation, real price of oil and exchange rate variables are estimated. Firstly, the benchmark GVAR model (BM) closely follows Dées et al. (DdPS, 2007) in which the country specific foreign variables, besides the oil price, are cross sectional weighted averages of domestic variables in the rest of the countries (star variables). The second GVAR model, also referred to as the extended GVAR model (EM), relaxes the restrictions imposed in the BM, and adds US variables to the set of foreign variables in the individual country models. This is in line with the US being treated as the globally dominant economy. The global dominance of the US implies that idiosyncratic shocks to the US economy have a non-negligible impact on potentially any country in the world while the impact of a small economy on the US is negligible. Thus, US variables effectively become dynamic common factors for other economies.

Empirical evidence is presented in support of relaxing the restrictions in the BM. The fraction of additional foreign US variables that have significant impact elasticity (contemporaneous effect) on the domestic counterpart in non-US country models is well above the nominal level of the tests. The dynamic properties of the two GVAR models are then compared by means of persistence profiles of a system wide shock to the cointegrating relationships and by means of generalized impulse response functions (GIRFs). We find that bootstrap confidence intervals for constructed GIRFs in the EM are, in the majority of cases, smaller than those in the BM. The choice of the appropriate GVAR model by allowing the US to be the dominant economy can be of major importance when assessing the magnitude of international economic linkages. For example in the case of negative 20bp shock to the US short term interest rate, we observe a significant response of Euro area short term interest rate in the EM (12bp decline in one year) whereas the response of Euro area short term interest rate in the BM is only 2bp and statistically insignificant beyond the first quarter.

The remainder of the paper is organized as follows. The next section describes the connection between IVAR models, equilibrium solution of DSGE models and the nature of the imposed restrictions. In Section 3 a nonstationary GVAR model is derived as an approximation to an infinite-dimensional VAR model, both considering and excluding the presence of a dominant economy.

[4]Weak and strong cross sectional dependence is formally defined in Chudik, Pesaran, and Tosetti (2011) and analyzed in the context of IVAR models by Chudik and Pesaran (2011).

Section 4 briefly discusses the asymptotic irrelevance of weights in stationary and non-stationary systems. The two versions of the GVAR model, BM and EM, are presented in Section 5. The final section offers some concluding remarks.

A brief word on notation: $\|\mathbf{A}\|_c$ denotes the maximum absolute column sum matrix norm of $\mathbf{A} \in \mathbb{M}^{n\times n}$, where $\mathbb{M}^{n\times n}$ is the space of real-entried $n \times n$ matrices. $\|\mathbf{A}\|_r$ is the absolute row-sum matrix norm of matrix $\mathbf{A}$.[5] $\|\mathbf{A}\| = \sqrt{\varrho\left(\mathbf{A}'\mathbf{A}\right)}$ is the spectral norm of matrix $\mathbf{A}$, $\varrho\left(\mathbf{A}\right)$ is the spectral radius of $\mathbf{A}$. Asymptotics $N, T \to \infty$ jointly are denoted by $N, T \xrightarrow{j} \infty$. $a_n = O(b_n)$ states the deterministic sequence a_n is at most of order b_n. Symbol $\xrightarrow{q.m.}$ represents the convergence in quadratic mean.

2 Infinite-Dimensional VAR as an Equilibrium Solution of N-country DSGE Models

DSGE models are becoming increasingly influential in contemporary macroeconomic policy making. These models are highly nonlinear in nature and therefore approximate solution techniques are employed to solve for equilibrium. The most common solution method of DSGE models is log-linearization around a steady state, which allows to characterize the equilibrium by a system of linear rational expectation equations. As it is discussed for example in Binder and Pesaran (1997), solution to a system of linear rational expectation equations takes form of a VAR model, in general. Hence the log-linear approximation to the equilibrium of a DSGE model is in essence a VAR model, such as the following reduced form system (abstracting from deterministic terms).

$$\mathbf{\Phi}\left(L\right)\mathbf{x}_t = \mathbf{D}\boldsymbol{\xi}_t, \tag{1}$$

where $\mathbf{\Phi}\left(L\right) = \left(\mathbf{I}_k - \sum_{\ell=1}^{p}\mathbf{\Phi}_\ell L^\ell\right)$ is a polynomial in L, L denotes the lag operator, $\mathbf{D}$ and $\{\mathbf{\Phi}_\ell\}_{\ell=1}^{p}$ are $k \times k$ dimensional matrices of coefficients, k is the number of endogenous variables and $\boldsymbol{\xi}_t$ is a vector of error terms.

In the closed economy models, a small number of key macroeconomic variables can be reliably estimated in a VAR. Owing to the mentioned curse of dimensionality, this is no longer true for multicountry models, where the dimension of $\mathbf{x}_t$ is considerably larger. In a multicountry set-up, the solution of DSGE models is not easily tractable and models are typically calibrated. Examples include SIGMA multicountry model of Federal Reserve Board (Erceg, Guerrieri, and Gust, 2006), or the GEM model developed by International Monetary Fund (see Laxton and Pesenti, 2003, and Bayoumi et al., 2004).

How to model a large set of endogenously determined variables without relying too much on a particular theoretical macroeconomic structure? This is most probably *the* fundamental problem in applied global macroeconomic modelling. Some restrictions must be imposed for the analysis of large systems. Bayesian estimations, where priors could come from a theoretical DSGE model were mentioned as one possibility. This paper follows another, novel approach, developed by Chudik and

[5] Maximum absolute column sum matrix norm and the maximum absolute row sum matrix norm are sometimes denoted in the literature as $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively.

Pesaran (2011), who propose restrictions that are binding *in the limit*, as the number of endogenous variables approaches infinity.

The first analyzed IVAR model in the next section assumes that coefficients corresponding to the foreign economies in the individual equations of the system (1) are of order $O\left(N^{-1}\right)$, where N denotes to the number of economies (excluding US). Such order restrictions are justified within the context of a micro-founded multicountry DSGE model by Chudik and Straub (2011) who studies the asymptotic properties of equilibrium for large N. In particular, this type of limiting restrictions corresponds to the equilibrium solution of N-country DSGE model where the mentioned small open economy framework is obtained in a limit as $N \to \infty$. Thus we refer to the first analyzed IVAR model as describing world consisting of small open economies.[6] The second IVAR model analyzed in the next section relaxes the first model by assuming that there is a globally dominant economy. In this case, as shown in the context of DSGE model by Chudik and Straub (2011), coefficients corresponding to the variables of the dominant economy in the individual equations of the equilibrium solution (1) are in general of order $O(1)$. The variables of the dominant economy here become possibly a dynamic common factors for the rest of the economies in the world.

3 GVAR as an Approximation to an Infinite-Dimensional VAR

Suppose there are $N+1$ countries indexed by $i = 0, 1, 2, ..., N$ and $k_i \geq 1$ endogenous variables belonging to country i, indexed by $s = 1, 2, ..., k_i$.[7] Let x_{ist} denote variable s of country i in period $t \in \{1, 2, .., T\}$ so that the country specific endogenous variables are given by vector $\mathbf{x}_{it} = (x_{i1t}, x_{i2t}, ..., x_{ik_it})'$. The $k \times 1$ dimensional vector $\mathbf{x}_t = (\mathbf{x}'_{0t}, \mathbf{x}'_{1t}, ..., \mathbf{x}'_{Nt})'$ collects the endogenous variables of all countries, where $k = \sum_{i=0}^{N} k_i$ denotes the total number of endogenous variables. Suppose that $\mathbf{x}_t$ is generated according to the following extended version of VAR model (1).

$$\mathbf{\Phi}(L)(\mathbf{x}_t - \boldsymbol{\delta}_0 - \boldsymbol{\delta}_1 t - \mathbf{\Gamma}_f \mathbf{f}_t) = \mathbf{D}\boldsymbol{\xi}_t, \tag{2}$$

where $\boldsymbol{\delta}_0 = \left(\boldsymbol{\delta}'_{00}, \boldsymbol{\delta}'_{01}, ..., \boldsymbol{\delta}'_{0N}\right)'$, $\boldsymbol{\delta}_1 = \left(\boldsymbol{\delta}'_{10}, \boldsymbol{\delta}'_{11}, ..., \boldsymbol{\delta}'_{1N}\right)'$, $\boldsymbol{\delta}'_{0i}$ and $\boldsymbol{\delta}'_{1i}$ are $k_i \times 1$ dimensional vectors of constants, $\mathbf{f}_t$ is a $m_f \times 1$ dimensional vector of unobserved common factors, $\mathbf{\Gamma}_f$ is a $k \times m_f$ dimensional matrix of factor loadings, $\boldsymbol{\xi}_t = \left(\boldsymbol{\xi}'_{0t}, \boldsymbol{\xi}'_{1t}, ..., \boldsymbol{\xi}'_{Nt}\right)'$, and $\boldsymbol{\xi}_{it}$ is $k_i \times 1$ dimensional cross-sectionally independent vector of country effects. This paper refers to model (2) also as an infinite-dimensional VAR, when the number of variables, k, is relatively large. Country-specific equations in the VAR model given by (2) can be written as

$$\sum_{j=0}^{N} \mathbf{\Phi}_{ij}(L)(\mathbf{x}_{jt} - \boldsymbol{\delta}_{0j} - \boldsymbol{\delta}_{1j} t - \mathbf{\Gamma}_{fj} \mathbf{f}_t) = \sum_{j=0}^{N} \mathbf{D}_{ij} \boldsymbol{\xi}_{jt} \text{ for } i = 0, 1, .., N, \tag{3}$$

[6] Note that it is not only the relative size of economies for the 'small open economy framework' to be justified asymptotically as an approximation to N-country model, which treats all countries endogenously, but, more importantly, the order of magnitude of the individual coefficients of export and import share matrices is the key. Chudik and Straub (2011) provides details.

[7] We start indexing countries with 0 in order to be compatible with the notation used in other GVAR articles in the literature.

where $[\mathbf{\Phi}_{ij}(L)]$, $i,j \in \{0,1,..,N\}$, represent partitioning of $\mathbf{\Phi}(L)$ into polynomials in $k_i \times k_j$ dimensional matrices, similarly $[\mathbf{D}_{ij}]$ represent partitioning of matrix $\mathbf{D}$ into $k_i \times k_j$ dimensional matrices, and the matrix of factor loadings is partitioned as $\mathbf{\Gamma}_f = \left(\mathbf{\Gamma}'_{f0}, \mathbf{\Gamma}'_{f1}, ..., \mathbf{\Gamma}'_{fN}\right)'$ with $\mathbf{\Gamma}_{fi}$ having dimension $k_i \times m_f$.

Several special cases of the complex VAR model (2) have been analyzed in the literature. Model (2) with $k_i = 1$ for $i = 0,1,..,N$, and $\mathbf{\Phi}(L) = \mathbf{D} = \mathbf{I}_{N+1}$, corresponds to a static representation of the (exact) dynamic factor model analyzed for example by Stock and Watson (2005). DdPS derive GVAR as an approximation to (2) for $\mathbf{\Phi}(L) = \mathbf{D} = \mathbf{I}_k$ and cross-sectionally independent, but serially correlated country specific effects $\{\boldsymbol{\xi}_{jt}\}$. VAR model (2) with further restrictions on coefficient matrices and without the unobserved common factors has been analyzed by Binder, Hsiao, and Pesaran (2005) for the large N and small T case. The focus of this paper is solely on large N and large T panels and on endogenous variables integrated of order one, $I(1)$ for short. This is accomplished by introducing the unit root properties in the unobserved processes $\mathbf{f}_t$ and $\boldsymbol{\xi}_t$, and, for the simplicity of exposition, all roots of $|\mathbf{\Phi}(L)| = 0$ are assumed to lie outside the unit circle.

System (2) models deviations of endogenous variables from common factors $\mathbf{f}_t$ and deterministic terms in a VAR model. Alternatively, unobserved common factors could be introduced in the residuals. Consider the following alternative data generating process (DGP)

$$\mathbf{\Phi}(L)(\mathbf{x}_t - \boldsymbol{\delta}_0 - \boldsymbol{\delta}_1 t) = \boldsymbol{\vartheta}_t, \tag{4}$$

where $\boldsymbol{\vartheta}_t$ is $k \times 1$ dimensional vector of error terms, given by

$$\boldsymbol{\vartheta}_t = \mathbf{\Gamma}_g \mathbf{g}_t + \mathbf{D}\boldsymbol{\xi}_t, \tag{5}$$

$\mathbf{g}_t$ is $m_g \times 1$ vector of unobserved common factors, and $\mathbf{\Gamma}_g$ is $k \times m_g$ dimensional matrix of factor loadings.[8] While it appears to be more conventional in the econometrics of panel data to introduce the unobserved common factors in the residuals, the choice between models (2) and (4) is to some extent arbitrary. Appendix B shows that models (4) and (2) are approximately the same models, but with possibly different numbers of unobserved common factors. Therefore focusing solely on either of the models bears no loss of generality, having treated the number of unobserved common factors as unknown. Hereafter, the focus will be solely on model (2), which is analytically more convenient. As it will become clear in what follows, this is because under certain assumptions about the coefficient matrices $\{\mathbf{\Phi}_\ell\}_{\ell=1}^p$ and $\mathbf{D}$, cross-sectional averages of the right hand side of (2) converge in quadratic mean to zero as $N \to \infty$ and the unobserved common factors $\mathbf{f}_t$ can be approximated by cross-sectional averages of the endogenous variables $\mathbf{x}_t$. A similar idea has been explored for the purpose of estimation and inference in large heterogenous panels with multifactor error structure by Pesaran (2006).

[8] See also Pesaran (2007) for a brief discussion of alternative residual serial correlation models with cross-section dependence.

3.1 Derivation of the GVAR in a World With Many Small Open Economies

Assume $\mathbf{x}_t$ is generated according to VAR model (2). In addition let $m_f \leq k^* < \infty$, where integer k^* does not depend on N, and let $\mathbf{W}_i = (\mathbf{W}_{i0}, \mathbf{W}_{i1}, ..., \mathbf{W}_{iN})'$ be any pre-determined $k \times k^*$ dimensional matrix of weights satisfying the following granularity conditions

$$\begin{aligned} \|\mathbf{W}_i\| &= O\left(N^{-\frac{1}{2}}\right), & (6) \\ \frac{\|\mathbf{W}_{ij}\|}{\|\mathbf{W}_i\|} &= O\left(N^{-\frac{1}{2}}\right). & (7) \end{aligned}$$

Define the following country specific vector of weighted averages

$$\mathbf{x}_{it}^* \equiv \mathbf{W}_i'\mathbf{x}_t = \sum_{j=0}^{N} \mathbf{W}_{ij}\mathbf{x}_{jt}. \tag{8}$$

One possible example of $\mathbf{W}_i$ is the matrix of weights defining cross-sectional averages.[9] The following assumptions on VAR model (2) are postulated.

ASSUMPTION 1 *Let $\mathbf{\Phi}_\ell = [\mathbf{\Phi}_{\ell,ij}]$, $i, j \in \{0, 1, .., N\}$, represent partitioning of matrix $\mathbf{\Phi}_\ell$ into $k_i \times k_j$ dimensional submatrices and define the $k \times k_i$ dimensional matrix $\mathbf{\Phi}_{\ell,-i} \equiv (\mathbf{\Phi}_{\ell,i0}, ..., \mathbf{\Phi}_{\ell,i,i-1}, \mathbf{0}_{k_i \times k_i}, \mathbf{\Phi}_{\ell,i,i+1}, ..., \mathbf{\Phi}_{\ell,iN})'$. It is assumed that*

$$\begin{aligned} \|\mathbf{\Phi}_{\ell,-i}\|_r &= O\left(N^{-1}\right), & (9) \\ \|\mathbf{D}\|_r \|\mathbf{D}\|_c &= O(1), & (10) \\ \mathbf{D}_{ii} &= \mathbf{I}_{k_i} \text{ (Normalization condition)}, & (11) \\ k_i &= O(1), & (12) \end{aligned}$$

for any $i \in \{0, 1, .., N\}$. Furthermore,

$$\sum_{\ell=0}^{\infty} \|\mathbf{R}_\ell\| = O(1), \tag{13}$$

[9] It is not required that $\mathbf{W}_{ii} = \mathbf{0}$ for convergence results presented in this section. Weights used in the estimation, however, satisfy also requirement $\mathbf{W}_{ii} = \mathbf{0}$ for $i \in \{0, .., N\}$.

where $\mathbf{R}_\ell$, *for* $\ell = 0, 1, 2, ...,$ *is top-left* $k \times k$ *submatrix of the matrix* $\mathbf{\Phi}^\ell$, [10]

$$\underset{kp\times kp}{\mathbf{\Phi}} = \begin{pmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \mathbf{\Phi}_3 & \cdots & \mathbf{\Phi}_p \\ \mathbf{I}_k & \mathbf{0}_k & \mathbf{0}_k & \cdots & \mathbf{0}_k \\ \mathbf{0}_k & \mathbf{I}_k & \mathbf{0}_k & & \mathbf{0}_k \\ \vdots & & \ddots & & \vdots \\ \mathbf{0}_k & \mathbf{0}_k & \cdots & \mathbf{I}_k & \mathbf{0}_k \end{pmatrix}.$$

ASSUMPTION 2 *The common factors* $\mathbf{f}_t$ *and country-specific effects* $\{\boldsymbol{\xi}_{it}\}$ *follow the processes*

$$\Delta\mathbf{f}_t = \mathbf{\Lambda}(L)\mathbf{v}_{ft}, \ \mathbf{v}_{ft} \sim IID\left(\mathbf{0}, \mathbf{\Omega}_{v_f}\right), \tag{15}$$

$$\Delta\boldsymbol{\xi}_{it} = \mathbf{\Psi}_i(L)\mathbf{v}_{it}, \ \mathbf{v}_{it} \sim IID\left(\mathbf{0}, \mathbf{\Omega}_{vi}\right), \tag{16}$$

where the polynomials

$$\mathbf{\Lambda}(L) = \mathbf{I}_{m_f} - \sum_{\ell=1}^{\infty}\mathbf{\Lambda}_\ell L^\ell \text{ and } \mathbf{\Psi}_i(L) = \mathbf{I}_{k_i} - \sum_{\ell=1}^{\infty}\mathbf{\Psi}_{i\ell}L^\ell \tag{17}$$

are absolute summable, $\mathbf{\Lambda}(L)$ *and* $\mathbf{\Psi}_i(L)$ *are invertible, and the variance of* $\Delta\mathbf{f}_t$ *and* $\Delta\boldsymbol{\xi}_{it}$ *is uniformly bounded in* N, *in particular*

$$\left\|\mathbf{\Omega}_{v_f}\right\| < K < \infty, \tag{18}$$

$$\max_{i\in\{0,1,..,N\}}\left\|\mathbf{\Omega}_{vi}\right\|_r < K < \infty, \tag{19}$$

$$\sum_{\ell=0}^{\infty}\max_{i\in\{0,1,..,N\}}\left\|\mathbf{\Psi}_{i\ell}\right\|_r < K < \infty. \tag{20}$$

ASSUMPTION 3 *Let* $\mathbf{y}_t \equiv \mathbf{x}_t - \boldsymbol{\delta}_0 - \boldsymbol{\delta}_1 t - \mathbf{\Gamma}_f\mathbf{f}_t$ *and* $\mathbf{u}_t \equiv \mathbf{D}\boldsymbol{\xi}_t$. *Define set* $\mathcal{S} \equiv \{2-p, ..., -1, 0\}$ *for* $p \geq 2$ *and* $\mathcal{S} \equiv \emptyset$ *for* $p = 1$. *It is assumed that*

$$\mathbf{y}_{1-p} \sim \left(\mathbf{0}, \mathbf{\Sigma}_{1-p}\right), \ \varrho\left(\mathbf{\Sigma}_{1-p}\right) = O(1), \tag{21}$$

$$\mathbf{u}_1 = \mathbf{R}(L)\Delta\mathbf{u}_1 - \left(\sum_{\ell=1}^{p}\mathbf{\Phi}_\ell - \mathbf{I}_k\right)\mathbf{y}_{1-p} - \sum_{s\in\mathcal{S}}\left(\sum_{\ell=1}^{1-s}\mathbf{\Phi}_\ell - \mathbf{I}_k\right)\Delta\mathbf{y}_{-s}, \tag{22}$$

where $\mathbf{R}(L) = \sum_{\ell=0}^{\infty}\mathbf{R}_\ell L^\ell$, $\mathbf{R}_\ell$ *is defined in Assumption 1. Furthermore,*

$$\Delta\mathbf{y}_\ell = \mathbf{R}(L)\Delta\mathbf{u}_\ell, \tag{23}$$

for $\ell \in \mathcal{S}$, *and* $\mathbf{y}_{1-p}$ *is independently distributed from* $\Delta\mathbf{y}_\ell$ *for* $\ell \in \mathcal{S}$ *and from* $\boldsymbol{\xi}_t$ *for any* $t \in$

[10] Alternatively, matrices $\mathbf{R}_\ell$ satisfy

$$\mathbf{R}_\ell = \sum_{h=1}^{p}\mathbf{\Phi}_h\mathbf{R}_{\ell-h} \tag{14}$$

with starting values $\mathbf{R}_0 = \mathbf{I}_k$, and $\mathbf{R}_\ell = \mathbf{0}_k$ for $\ell < 0$.

$\{1, .., T\}$.

ASSUMPTION 4 *Matrix $\mathbf{\Gamma}_{fi}^*$ has full column rank.*

Remark 1 *Assumption 1 allows for a general pattern of weak cross-sectional dependence of $\mathbf{u}_t$ (including any form of commonly used spatial dependence in the literature, see Chudik, Pesaran, and Tosetti, 2011), but excludes additional common factors besides the vector $\mathbf{f}_t$.*

Remark 2 *Condition (13) of Assumption 1 implies that all roots of $|\mathbf{\Phi}(L)| = 0$ lie outside the unit circle and a point on the unit circle is not a limit point of the sequence of roots of $|\mathbf{\Phi}(L)| = 0$ as $N \to \infty$. Hence, for each $N \in \mathbb{N}$, $\mathbf{\Phi}(L)$ is invertible and $\mathbf{\Phi}^{-1}(L) = \mathbf{R}(L) = \sum_{\ell=0}^{\infty} \mathbf{R}_\ell L^\ell$.*

Remark 3 *Assumptions 1, 2 and Assumption 3 on starting values imply that endogenous variables are $I(1)$.*

Remark 4 *Assumption 4 is required for approximation of unobserved common factors f_t by a linear combination of observed cross sectional averages x_{it}^* and deterministic terms.*

Vector $\mathbf{y}_{it}^* = \mathbf{W}_i' (\mathbf{x}_t - \boldsymbol{\delta}_0 - \boldsymbol{\delta}_1 t - \mathbf{\Gamma}_f \mathbf{f}_t)$ can be written as

$$\mathbf{y}_{it}^* = \sum_{\ell=2-p}^{t} \Delta \mathbf{y}_{i\ell}^* + \mathbf{W}_i' \mathbf{y}_{1-p} \ . \tag{24}$$

Starting values $\mathbf{y}_{1-p}$ are cross sectionally weakly dependent (see Proposition 2.1 of Chudik, Pesaran, and Tosetti, 2011) and distributed with mean $\mathbf{0}$ under Assumption 3. Therefore it follows that

$$\mathbf{W}_i' \mathbf{y}_{1-p} \overset{q.m.}{\to} \mathbf{0}_{k^*}, \tag{25}$$

as $N \to \infty$, for any weight matrix $\mathbf{W}_i$ satisfying condition (6) only.[11]

Lemma 1 *Consider model (2) and let Assumptions 1-3 hold. Then for any weight matrix $\mathbf{W}_i$ satisfying condition (6) only,*

$$\left\| Var\left(\sum_{\ell=2-p}^{t} \Delta \mathbf{y}_{i\ell}^* \right) \right\| = O\left(\frac{T}{N} \right),$$

where $1 \leq t \leq T$, $\mathbf{y}_{i\ell}^* = \sum_{j=0}^{N} \mathbf{W}_{ij} \mathbf{y}_{j\ell}$ and $\mathbf{y}_{j\ell}$ is defined in Assumption 3.

[11] Using the Rayleigh-Ritz theorem (see Horn and Johnson, 1985, 1985, p. 176),

$$\begin{aligned} \left\| Var\left(\mathbf{W}_i' \mathbf{y}_{1-p}\right) \right\| &= \left\| \mathbf{W}_i' \mathbf{\Sigma}_{1-p} \mathbf{W}_i \right\|, \\ &\leq \varrho\left(\mathbf{\Sigma}_{1-p}\right) \left\| \mathbf{W}_i' \mathbf{W}_i \right\| = O(1) \left\| \mathbf{W}_i' \mathbf{W}_i \right\|, \end{aligned} \tag{26}$$

where $\varrho(\mathbf{\Sigma}_{1-p}) = O(1)$ by condition (21) of Assumption 3. Self-adjoint and submultiplicative properties of the spectral norm establish that $\|\mathbf{W}_i' \mathbf{W}_i\| \leq \|\mathbf{W}_i\|^2$. But $\|\mathbf{W}_i\|^2 = O(N^{-1})$ according to granularity condition (6). It follows that $\|Var(\mathbf{W}_i' \mathbf{y}_{1-p})\| = O(N^{-1})$.

Proof of Lemma 1 is relegated to Appendix.

Lemma 1 and equation (25) imply that for any increasing integer value function $T(N) : \mathbb{N} \to \mathbb{N}$ satisfying

$$\lim_{N\to\infty} \frac{T(N)}{N} = 0, \tag{27}$$

the following equality holds

$$\lim_{N\to\infty} \sup_{1\leq t\leq T(N)} \|Var(\mathbf{y}_{it}^{*})\| = 0. \tag{28}$$

Since also $E(\mathbf{y}_{it}^{*}) = 0$, we say that

$$\mathbf{y}_{it}^{*} \overset{q.m.}{\to} \mathbf{0} \tag{29}$$

uniformly in $t \in \{1, 2, ..., T(N)\}$ as $N, T \overset{j}{\to} \infty$, such that $T/N \to 0$, or as $N \to \infty$ followed by $T \to \infty$. Equation (28) establishes that it is possible to approximate the unobserved common factors $\mathbf{f}_t$ by a linear combination of $\{\mathbf{x}_{it}^{*}, 1, t\}$ if also Assumption 4 holds.[12] Multiplying the equation (28) by matrix $\left(\mathbf{\Gamma}_{fi}^{*\prime}\mathbf{\Gamma}_{fi}^{*}\right)^{-1}\mathbf{\Gamma}_{fi}^{*\prime}$ from the left yields

$$\mathbf{f}_t - \left(\mathbf{\Gamma}_{fi}^{*\prime}\mathbf{\Gamma}_{fi}^{*}\right)^{-1}\mathbf{\Gamma}_{fi}^{*\prime}\left(\mathbf{x}_{it}^{*} - \boldsymbol{\delta}_{0i}^{*} - \boldsymbol{\delta}_{1i}^{*}t\right) \overset{q.m.}{\to} \mathbf{0}, \tag{30}$$

uniformly in t under the same asymptotics. Equation (30) holds for any weights $\mathbf{W}_i$ satisfying granularity condition (6) only. Notice that under Assumption 1, matrices $\{\mathbf{\Phi}_{\ell,-i}\}_{\ell=1}^{p}$ satisfy the following inequality, see Bernstein (2005, p. 369, Fact 9.8.15).

$$\|\mathbf{\Phi}_{\ell,-i}\|^2 \leq \|\mathbf{\Phi}_{\ell,-i}\|_r \|\mathbf{\Phi}_{\ell,-i}\|_c = O\left(N^{-1}\right), \tag{31}$$

where $\|\mathbf{\Phi}_{\ell,-i}\|_c \leq k\|\mathbf{\Phi}_{\ell,-i}\|_r = O(1)$ and $\|\mathbf{\Phi}_{\ell,-i}\|_r = O\left(N^{-1}\right)$ by Assumption 1. Therefore it follows from (30) and (31) that

$$\mathbf{\Phi}_{\ell,-i}\mathbf{y}_t \overset{q.m.}{\to} \mathbf{0}_{k_i}, \tag{32}$$

uniformly in t as $N, T \overset{j}{\to} \infty$, such that $T/N \to 0$ (or under the sequential asymptotics $N \to \infty$ followed by $T \to \infty$), and the country-specific equation (3) reduces to

$$\mathbf{\Phi}_{ii}(L)\left(\mathbf{x}_{jt} - \boldsymbol{\delta}_{0,j} - \boldsymbol{\delta}_{1,j}t - \mathbf{\Gamma}_{f,j}\mathbf{f}_t\right) - \mathbf{u}_{it} \overset{q.m.}{\to} \mathbf{0}_{k_i}, \tag{33}$$

uniformly in t under the same asymptotics. The process $\Delta\mathbf{u}_{it} = \sum_{j=0}^{N}\mathbf{D}_{ij}\Delta\boldsymbol{\xi}_{jt}$ is a stationary weakly cross-sectionally dependent process. Using the Wold decomposition theorem, $\Delta\mathbf{u}_{it}$ can be written as

$$\Delta\mathbf{u}_{it} = \mathbf{Q}_i(L)\boldsymbol{\epsilon}_{it}, \tag{34}$$

[12] Without Assumption 4, $\{\mathbf{x}_{it}^{*}, \boldsymbol{\delta}_{0i}^{*}, \boldsymbol{\delta}_{1i}^{*}t\}$ would not suffice to approximate the unobserved common factors. Note also that although the granularity condition (7) is not required for result (30), it establishes that $\|\mathbf{W}_i\|_c$ has a nonzero limit as $N \to \infty$, which is a necessary condition for Assumption 4 to hold, having bounded country-specific factor loadings $\mathbf{\Gamma}_{fi}$ in N.

where $\boldsymbol{\epsilon}_{it}$ is serially uncorrelated, but in general weakly cross-sectionally dependent. Using the following approximation

$$\mathbf{H}_i(L, p_i) \approx (1 - L)\,\mathbf{Q}_i^{-1}(L)\,\boldsymbol{\Phi}_{ii}(L), \tag{35}$$

individual VAR models are obtained as

$$\mathbf{H}_i(L, p_i)(\mathbf{x}_{it} - \boldsymbol{\delta}_{0i} - \boldsymbol{\delta}_{1i}t - \boldsymbol{\Gamma}_{fi}\mathbf{f}_t) - \boldsymbol{\epsilon}_{it} \approx \mathbf{0}_{k_i}. \tag{36}$$

Country-specific models (36) with common factors proxied by (30) with $\mathbf{x}_{it}^*$ entering as weakly exogenous can be estimated separately by techniques developed by Harbo et al. (1998) and Pesaran, Shin, and Smith (2000).[13] After the individual VARX* models are estimated, they can be combined in a GVAR model featuring all endogenous variables. This idea was originally introduced by PSW.

3.2 Derivation of a GVAR Model Featuring a Dominant Economy

This section derives GVAR as an approximation to the infinite dimensional VAR model (2) with country 0 being dominant. For clarity of exposition and in order to focus on the implication of a dominant country, it is assumed without loss of generality that $p = 1$.

ASSUMPTION 5 *Define $k \times k_0$ dimensional matrix $\boldsymbol{\Phi}_0 \equiv (\boldsymbol{\Phi}_{00}', ..., \boldsymbol{\Phi}_{N0}')'$, a $k \times k_0$ dimensional selection matrix $\mathbf{S}_0 = (\mathbf{I}_{k_0}, \mathbf{0}, ..., \mathbf{0})'$ and $k \times k$ dimensional matrix $\boldsymbol{\Phi}_w \equiv \boldsymbol{\Phi} - \boldsymbol{\Phi}_0\mathbf{S}_0'$. Similarly, define $\mathbf{D}_w \equiv \mathbf{D} - \mathbf{D}_0\mathbf{S}_0'$, where $\mathbf{D}_0 \equiv (\mathbf{D}_{00}', ..., \mathbf{D}_{N0}')'$. It is assumed that*

$$\|\boldsymbol{\Phi}_{w,-i}\|_r = O\left(N^{-1}\right), \tag{37}$$

$$\|\mathbf{D}_{w,-i}\|_r = O\left(N^{-1}\right), \tag{38}$$

for all i, where $\boldsymbol{\Phi}_{w,-i} = (\mathbf{0}, \boldsymbol{\Phi}_{i1}, ..., \boldsymbol{\Phi}_{i,i-1}, \mathbf{0}, \boldsymbol{\Phi}_{i,i+1}, ..., \boldsymbol{\Phi}_{iN})'$ and $\mathbf{D}_{w,-i} = (\mathbf{0}, \mathbf{D}_{i1}..., \mathbf{D}_{i,i-1}, \mathbf{0}, \mathbf{D}_{i,i+1}, ..., \mathbf{D}_{iN})'$. Furthermore,

$$\|\boldsymbol{\Phi}\|_r < \rho < 1. \tag{39}$$

ASSUMPTION 6 *Matrix*

$$\underset{(k_0+k^*)\times(k_0+m_f)}{\mathbf{M}_i} = \begin{pmatrix} \mathbf{D}_{00} & \boldsymbol{\Gamma}_{f0} \\ \mathbf{W}_i'\mathbf{D}_0 & \boldsymbol{\Gamma}_{fi}^* \end{pmatrix} \tag{40}$$

has full column rank for any $i \in \{0, 1, ..., N\}$.

Remark 5 *Under Assumption 5 all roots of $\boldsymbol{\Phi}(L) = \mathbf{I}_k - \boldsymbol{\Phi}L$ lie outside the unit circle and no point on the unit circle is a limit point of the sequence of eigenvalues of $\boldsymbol{\Phi}$ as $N \to \infty$. Note also that under Assumption 5 $\|\boldsymbol{\Phi}\|_c = O(N)$. As discussed in Chudik and Pesaran (2012), the presence of a dominant column in the matrix $\boldsymbol{\Phi}$ implies $\{\mathbf{y}_{it}\}$ is cross sectionally strongly dependent.*

[13] Note that these techniques were developed under the assumption of no cointegration among star variables, which is not likely to hold. Extension of these techniques to allow for cointegration among star variables is currently, however, an open research question, which will not be pursued here.

Remark 6 *Assumptions 2, 3 and 5 imply* $\Delta \mathbf{y}_t = \sum_{\ell=t}^{\infty} \mathbf{\Phi}^{\ell} \Delta \mathbf{u}_{t-\ell}$ *is stationary.*

Multiplying VAR model (2) by $\mathbf{W}_i'$ from the left and taking the first differences under Assumptions 2, 3 and 5 yields

$$\begin{aligned} \Delta \mathbf{y}_{it}^* &= \sum_{\ell=0}^{\infty} \mathbf{W}_i' \mathbf{\Phi}^{\ell} \mathbf{D} \Delta \boldsymbol{\xi}_{t-\ell}, \\ &= \sum_{\ell=0}^{\infty} \mathbf{W}_i' \mathbf{\Phi}^{\ell} \mathbf{D}_0 \Delta \boldsymbol{\xi}_{0,t-\ell} + \sum_{\ell=0}^{\infty} \mathbf{W}_i' \mathbf{\Phi}^{\ell} \mathbf{D}_w \Delta \boldsymbol{\xi}_{t-\ell}. \end{aligned} \tag{41}$$

But $\sum_{\ell=0}^{\infty} \mathbf{W}_i' \mathbf{\Phi}^{\ell} \mathbf{D}_w \Delta \boldsymbol{\xi}_t \overset{q.m.}{\to} \mathbf{0}$ as $N \to \infty$ by Lemma 3 in the Appendix. Hence

$$\Delta \mathbf{y}_{it}^* - \mathbf{\Theta}\left(\mathbf{W}_i, L\right) \Delta \boldsymbol{\xi}_{0t} \overset{q.m.}{\to} \mathbf{0}, \tag{42}$$

as $N \to \infty$, where $\mathbf{\Theta}\left(\mathbf{W}_i, L\right) = \sum_{\ell=0}^{\infty} \mathbf{W}_i' \mathbf{\Phi}^{\ell} \mathbf{D}_0 L^{\ell}$. Similarly, we have by Lemma 3,

$$\Delta \mathbf{y}_{0t} - \mathbf{\Theta}\left(\mathbf{S}_0, L\right) \Delta \boldsymbol{\xi}_{0t} = \sum_{\ell=0}^{\infty} \mathbf{S}_0' \mathbf{\Phi}^{\ell} \mathbf{D}_w \Delta \boldsymbol{\xi}_{t-\ell} \overset{q.m.}{\to} \mathbf{0}, \tag{43}$$

as $N \to \infty$, where $\mathbf{\Theta}\left(\mathbf{S}_0, L\right) = \sum_{\ell=0}^{\infty} \mathbf{S}_0' \mathbf{\Phi}^{\ell} \mathbf{D}_0 L^{\ell}$. For simplicity of exposition, the focus of this subsection is on the convergence at the point in time t, and $N \to \infty$. Equations (42) and (43), assuming Assumption 6 also holds, establish

$$\Delta \left[\begin{pmatrix} \mathbf{x}_{0t} \\ \mathbf{x}_{it}^* \end{pmatrix} - \begin{pmatrix} \boldsymbol{\delta}_{01} \\ \boldsymbol{\delta}_{i1}^* \end{pmatrix} - \begin{pmatrix} \boldsymbol{\delta}_{02} \\ \boldsymbol{\delta}_{i2}^* \end{pmatrix} t \right] - \mathbf{A}_i\left(L\right) \mathbf{M}_i \begin{pmatrix} \mathbf{v}_{0t} \\ \mathbf{v}_{ft} \end{pmatrix} \overset{q.m.}{\to} 0, \tag{44}$$

where $\mathbf{A}_i\left(L\right) = \mathbf{I}_{k_0+k^*} - \sum_{\ell=1}^{\infty} \mathbf{A}_{i\ell} L^{\ell}$ is an absolute summable polynomial that satisfies

$$\mathbf{A}_i\left(L\right) \mathbf{M}_i = \begin{pmatrix} \mathbf{\Theta}\left(\mathbf{S}_0, L\right) & \mathbf{\Gamma}_{f0} \\ \mathbf{\Theta}\left(\mathbf{W}_i, L\right) & \mathbf{\Gamma}_{fi}^* \end{pmatrix} \begin{pmatrix} \mathbf{\Psi}_0\left(L\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}\left(L\right) \end{pmatrix}. \tag{45}$$

Equation (44) has two important implications. First, it establishes that the unobserved common factor $\mathbf{f}_t$ and the dynamic common factor $\boldsymbol{\xi}_{0t}$ can be approximated using $\{\mathbf{x}_{0t}, \mathbf{x}_{it}^*\}$, their lags, and deterministic terms. Assuming

$$\mathbf{E}_i\left(L, h_i\right) \approx \Delta \mathbf{A}_i^{-1}\left(L\right), \tag{46}$$

it follows from (44) that

$$\begin{pmatrix} \mathbf{v}_{0t} \\ \mathbf{v}_{ft} \end{pmatrix} \approx \left(\mathbf{M}_i' \mathbf{M}_i\right)^{-1} \mathbf{M}_i' \mathbf{E}_i\left(L, h_i\right) \left[\begin{pmatrix} \mathbf{x}_{0t} \\ \mathbf{x}_{it}^* \end{pmatrix} - \begin{pmatrix} \boldsymbol{\delta}_{01} \\ \boldsymbol{\delta}_{i1}^* \end{pmatrix} - \begin{pmatrix} \boldsymbol{\delta}_{02} \\ \boldsymbol{\delta}_{i2}^* \end{pmatrix} t \right]. \tag{47}$$

Observe that the set of (dynamic) common factors $\{\mathbf{f}_t, \boldsymbol{\xi}_{0t}\}$ can be successfully approximated only if $\mathbf{M}_i$ has full column rank (i.e. Assumption 6 holds). The number of unobserved common factors in $\mathbf{f}_t$ not exceeding the number of star variables, $m_f \leq k^*$, is therefore a necessary condition for

Assumption 6 to hold. In the case where $m_f > k^*$, the country-specific models cannot be consistently estimated from the VARX* models developed below. The second implication of result (44) is that the variables corresponding to the dominant group, $\mathbf{x}_{0t}$ and $\mathbf{x}^*_{0t}$, are generally endogenously determined. Rewriting (44) for $i = 0$ yields a model for the dominant country given by

$$\mathbf{E}_0\left(L, h_i\right)\left(\mathbf{z}_{0t} - \mathbf{d}_{01} - \mathbf{d}_{02}t\right) \approx \boldsymbol{\nu}_t, \tag{48}$$

where $\mathbf{z}_{0t} = \left(\mathbf{x}'_{0t}, \mathbf{x}^{*\prime}_{0t}\right)'$, $\mathbf{d}_{01} = \left(\boldsymbol{\delta}'_{01}, \boldsymbol{\delta}^{*\prime}_{01}\right)'$, $\mathbf{d}_{02} = \left(\boldsymbol{\delta}'_{02}, \boldsymbol{\delta}^{*\prime}_{02}\right)'$ and $\boldsymbol{\nu}_t = \mathbf{M}_i\left(\mathbf{v}'_{0t}, \mathbf{v}'_{ft}\right)'$.

Equations for the remaining countries are now easily derived. Taking first differences of the country-specific equations (3) yields

$$\Delta\mathbf{y}_{it} = \boldsymbol{\Phi}_{ii}\Delta\mathbf{y}_{i,t-1} + \boldsymbol{\Phi}'_{w,-i}\Delta\mathbf{y}_{t-1} + \boldsymbol{\Phi}_{i0}\Delta\mathbf{y}_{0,t-1} + \mathbf{D}_{i0}\Delta\boldsymbol{\xi}_{0t} + \Delta\boldsymbol{\xi}_{it} + \mathbf{D}'_{w,-i}\Delta\boldsymbol{\xi}_t\ . \tag{49}$$

Lemma 3 implies

$$\boldsymbol{\Phi}_{i0}\Delta\mathbf{y}_{0,t-1} - \boldsymbol{\Phi}_{i0}\boldsymbol{\Theta}\left(\mathbf{S}_0, L\right)\Delta\boldsymbol{\xi}_{0,t-1} \overset{q.m.}{\to} \mathbf{0}_{k_i}, \tag{50}$$

$$\boldsymbol{\Phi}'_{w,-i}\Delta\mathbf{y}_{t-1} - \boldsymbol{\Theta}\left(\boldsymbol{\Phi}'_{w,-i}, L\right)\Delta\boldsymbol{\xi}_{0,t-1} \overset{q.m.}{\to} \mathbf{0}_{k_i}, \tag{51}$$

as $N \to \infty$, where the second result follows from Lemma 3 by noting that $\left\|\boldsymbol{\Phi}_{w,-i}\right\|_r = O\left(N^{-1}\right)$ under Assumption 5. Furthermore,

$$\mathbf{D}'_{w,-i}\Delta\boldsymbol{\xi}_t \overset{q.m.}{\to} \mathbf{0}_{k_i}, \tag{52}$$

as $N \to \infty$, since

$$\left\|Var\left(\mathbf{D}'_{w,-1}\Delta\boldsymbol{\xi}_t\right)\right\|_r \leq \left\|\mathbf{D}_{w,-i}\right\|_r \left\|\mathbf{D}'_{w,-i}\right\|_c \left\|Var\left(\Delta\boldsymbol{\xi}_t\right)\right\|_r = O\left(N^{-1}\right), \tag{53}$$

where $\left\|Var\left(\Delta\boldsymbol{\xi}_t\right)\right\|_r \leq \max_i \left\|Var\left(\Delta\boldsymbol{\xi}_{it}\right)\right\|_r = O\left(1\right)$ by Assumption 2, and $E\left(\mathbf{D}'_{w,-i}\Delta\boldsymbol{\xi}_t\right) = \mathbf{0}_{k_i}$. Substituting equations (50)-(52) back into (49) establishes

$$\left(\mathbf{I}_{k_i} - \boldsymbol{\Phi}_{ii}L\right)\left(1-L\right)\left(\mathbf{x}_{it} - \boldsymbol{\delta}_{0i} - \boldsymbol{\delta}_{1i}t\right) - \mathbf{C}_i\left(L\right)\begin{pmatrix}\mathbf{v}_{0t}\\ \mathbf{v}_{ft}\end{pmatrix} - \Delta\boldsymbol{\xi}_{it} \overset{q.m.}{\to} \mathbf{0}_{k_i}, \tag{54}$$

as $N \to \infty$, where

$$\mathbf{C}_i\left(L\right) = \left(\ \left[\mathbf{D}_{i0} + \boldsymbol{\Theta}\left(\boldsymbol{\Phi}'_{w,-i} + \mathbf{S}_0\boldsymbol{\Phi}'_{i0}, L\right)L\right]\boldsymbol{\Psi}_i\left(L\right),\quad \left(\mathbf{I}_{k_i} - \boldsymbol{\Phi}_{ii}L\right)\boldsymbol{\Lambda}\left(L\right)\ \right).$$

Using (47), the unobserved vector of innovations for the common factors, $\left(\mathbf{v}'_{0t}, \mathbf{v}'_{ft}\right)'$, can be approximated by observable variables $\mathbf{x}_{0t}$, $\mathbf{x}^*_{it}$, their lags and deterministic terms. Substituting (47) into (54) and assuming

$$\begin{aligned}\mathbf{G}_i\left(L, p_i\right) &\approx \left(1-L\right)\boldsymbol{\Psi}_i^{-1}\left(L\right)\left(\mathbf{I}_{k_i} - \boldsymbol{\Phi}_{ii}L\right), &(55)\\ \Big(\underset{k_i\times k_0}{\mathbf{B}_{0i}\left(L, r_{0i}\right)}, \underset{k_i\times k^*}{\mathbf{B}_{1i}\left(L, r_{1i}\right)}\Big) &\approx \boldsymbol{\Psi}_i^{-1}\left(L\right)\mathbf{C}_i\left(L\right)\left(\mathbf{M}'_i\mathbf{M}_i\right)^{-1}\mathbf{M}'_i\mathbf{A}_i^{-1}\left(L\right)\left(1-L\right), &(56)\end{aligned}$$

the following country-specific VARX* models are obtained for $i > 0$,

$$\mathbf{G}_i(L, p_i)\mathbf{x}_{it} - \overline{\boldsymbol{\delta}}_{0i} - \overline{\boldsymbol{\delta}}_{0i}t - \mathbf{B}_{0i}(L, r_{0i})\mathbf{x}_{0t} - \mathbf{B}_{1i}(L, r_{1i})\mathbf{x}_{it}^* \approx \mathbf{v}_{it}. \tag{57}$$

Country-specific models (57) for $i = 1, .., N$ and the model (48) for a dominant country can be consistently estimated separately. Observe that there are k endogenous variables, but $k + k^*$ equations given in (57) and (48). Therefore, in order to construct a GVAR model featuring k endogenous variables in a coherent manner for the purpose of forecasting or impulse-response analysis, the conditional model for $\mathbf{x}_{0t}$ from (48) can be combined with country models (57) to form a GVAR model featuring all endogenous variables.

The main implications for empirical modelling are summarized as follows. In the presence of a dominant country 0, individual models for countries $i > 0$ need to be augmented by star variables $\mathbf{x}_{it}^*$ as well as variables $\mathbf{x}_{0t}$. The model for dominant economy $i = 0$ is a separate model and, without further restrictions, $\mathbf{x}_{0t}$ should generally be treated endogenously with $\mathbf{x}_{0t}^*$.

4 Dimensions of the Panel and Asymptotic Irrelevance of Weights

The selection of country-specific weights $\mathbf{W}_i$ defined in Section 3.1, as well as weights used in the analysis of stationary infinite-dimensional VAR models in Chudik and Pesaran (2011), are asymptotically irrelevant under the proper relative rates of convergence for N and T dimensions of the panel as long as they satisfy the stated minimal requirements.

In the case of stationary systems, any joint or sequential convergence for $N, T \to \infty$ could be assumed for de-coupling of a group from the system (uniformly in $t \in \{1, 2, .., T(N)\}$), once conditioned on all common factors. This is because for any weakly dependent stationary process $\boldsymbol{\zeta}_t$ with zero mean, absolute summable autocovariances, and $\varrho[Var(\boldsymbol{\zeta}_t)] = O(1)$, we have

$$\left\| Var\left(\mathbf{W}'\boldsymbol{\zeta}_t\right)\right\| = \left\|\mathbf{W}' Var(\boldsymbol{\zeta}_t)\mathbf{W}\right\| \le \varrho[Var(\boldsymbol{\zeta}_t)]\left\|\mathbf{W}\right\|^2 = O\left(N^{-1}\right), \tag{58}$$

where $\|\mathbf{W}\|^2 = O\left(N^{-1}\right)$ by granularity condition (6). Equation (58) implies[14]

$$\lim_{N\to\infty} \sup_{1\le t\le T(N)} Var\left(\mathbf{W}'\boldsymbol{\zeta}_t\right) = 0 \tag{59}$$

for any increasing integer-valued function $T(N)$. For the purpose of estimation and inference in stationary IVARs, Chudik and Pesaran (2011) show that the relevant asymptotics are $T, N \xrightarrow{j} \infty$ with $T/N \to \kappa < \infty$. This means that the time dimension should not go to infinity at a faster rate than the number of groups. Theoretical results developed in Chudik and Pesaran (2011) are thus suitable for balanced panels, or for panels where N is large relative to T, and T is not very small.

In the case of cointegrated $I(1)$ systems, a stronger requirement on the dimensions of the panel is needed for the asymptotic irrelevance of chosen weights. $T/N \to 0$ as $N, T \xrightarrow{j} \infty$ is required

[14]Note that the slightly weaker condition $\varrho[Var(\boldsymbol{\zeta}_t)] = o(N)$ is sufficient for (59) to hold.

in Section 3.1 in order to approximate the unobserved factors $\mathbf{f}_t$ (uniformly in t) using observable averages. Again, the intuition is simple. For any cross-sectionally weakly dependent $I(1)$ process $\boldsymbol{\omega}_t = \sum_{\ell=0}^{t} \boldsymbol{\zeta}_\ell$, we have

$$\left\| Var\left(\mathbf{W}'\boldsymbol{\omega}_t\right)\right\| = \left\| Var\left(\mathbf{W}'\sum_{\ell=0}^{t}\boldsymbol{\zeta}_\ell\right)\right\| \leq \left\|\mathbf{W}\right\|^2 \left\| Var\left(\sum_{\ell=0}^{t}\boldsymbol{\zeta}_\ell\right)\right\| \leq tO\left(N^{-1}\right), \tag{60}$$

which proves that

$$\lim_{N\to\infty} \sup_{1\leq t\leq T(N)} \left\| Var\left(\mathbf{W}'\boldsymbol{\omega}_t\right)\right\| = 0, \tag{61}$$

for any increasing integer-valued function $T(N)$ satisfying

$$\lim_{N\to\infty} \frac{T(N)}{N} = 0.$$

That is, if joint asymptotics $N, T \xrightarrow{j} \infty$ are considered ($T \to \infty$ is needed for consistent estimation), then $T/N \to 0$ for equation (61) to hold. Unit root and cointegration properties of $\mathbf{x}_t$ in Section 3 are accommodated for the simplicity of exposition by allowing the unobserved factors $\mathbf{f}_t$ and/or the country-specific effects, $\boldsymbol{\xi}_{it}$, to have unit roots and all roots of polynomial $\boldsymbol{\Phi}(L)$ lie outside the unit circle. Thus the cross-sectional cointegration is *a priori* restricted to coming through the common factors only. Appendix A shows, in the context of a more general nonstationary infinite-dimensional VAR(1) model, that this simplifying assumption is not restrictive under the postulated restrictions on coefficients and asymptotics $N, T \xrightarrow{j} \infty$ such that $T/N \to 0$.

Since the group dimension (number of countries) needs to be large relative to the time dimension of the panel in the case of $I(1)$ variables (as opposed to stationary variables) for the above *uniform* convergence results to hold, it is in general more likely that a purely ad-hoc selection of weights is an empirically irrelevant issue in the context of modelling stationary systems (as opposed to nonstationary systems).[15] On the other hand, the main disadvantage of modelling variables that have been transformed to induce stationarity is the potential bias resulting from the omitted error-correction terms.

Economic theory does not have a unique answer regarding the selection of weights. For example, the cointegrating vector for relative prices and relative productivities across multiple economies typically depends on the joint assumptions of the macroeconomic model under consideration.

5 The GVAR Model in which the US is Treated as the Globally Dominant Economy

This section develops two versions of a GVAR model of the global economy. Country and time coverage is the same as in DdPS. 33 countries feature in the analysis, representing 90% of the world's

[15] Some variables, such as equity and bond prices tend to move very closely across economies and therefore the selection of different weights is unlikely to matter much here.

output, where 8 of the 11 countries that originally joined the Euro on January 1, 1999 are grouped together and the remaining countries are modelled separately. Data is quarterly and individual country VARX* models were estimated over the period 1979(Q2)-2003(Q4). Thus $N = 26$ and $T = 98$. Detailed explanations of the data sources and construction of the data can be found in DdPS (2007) .

The selection of variables is one of the most important modelling choices. The GVAR model developed in DdPS includes (depending on the availability) 6 endogenous variables in the individual country models: real output (y_{it}), the rate of inflation ($\pi_{it} = \Delta p_{it}$), the real equity prices (q_{it}), the short term interest rate (ρ^s_{it}), a long rate of interest (ρ^L_{it}) and an exchange rate variable[16].

The selection of variables must be balanced with the availability of the data. Treating the US as a globally dominant economy comes at an additional cost in terms of degrees of freedom. Consider country models (57). If the number of lags are restricted to $p_i \leq 2$ (for domestic endogenous variables), $r_{0i} = 1$ (for weakly exogenous US variables) and $r_{1i} = 1$ (for weakly exogenous cross-sectional averages), then the number of parameters per equation to be estimated in country models is 36.[17] This is clearly not reliable and therefore the number of endogenous variables must be reduced in order to increase the degrees of freedom. In particular, long-term interest rates and real equity prices were omitted from the model. Modelling choices regarding the exchange rate variable follow those found in the DdPS model.

Two models were constructed for this paper. The first is the benchmark GVAR model (BM), which is similar to the DdPS version of GVAR, but without the real stock prices and long term interest rates. Where available, endogenous (domestic) variables in country models were

$$\mathbf{x}_{it} = (p_{it} - e_{it}, \pi_{it}, y_{it}, \rho^s_{it})', \text{ for } i \geq 1 \tag{62}$$

and

$$\mathbf{x}_{0t} = (\pi_{0t}, y_{0t}, \rho^s_{0t}, p^o_t)' \text{ (US model).} \tag{63}$$

Oil prices are an observed common factor and they were therefore included as endogenous variable in the US model and as a weakly exogenous variable in the remaining country models. The vector of weakly exogenous foreign variables in the individual country models is

$$(y^*_{it}, \pi^*_{it}, \rho^{s*}_{it}, p^o_t), \text{ for } i > 0 \text{ (BM only)} \tag{64}$$

and for the US

$$(y^*_{0t}, \pi^*_{0t}). \tag{65}$$

Variables $\rho^{s*}_{0t}, e^*_{0t} - p^*_{0t}$ were not included as weakly exogenous in the benchmark US model as the

[16] DdPS employed domestic consumer price level converted into US$, variable $e_{it} - p_{it}$, while Dées, Holly, Pesaran, and Smith (2007) used the trade-weighted real effective exchange rate, $\overline{re}_{it} = (e_{it} - p_{it}) - (e^*_{it} - p^*_{it})$. The price of oil ($p^o_t$) is treated as an observed endogenous common factor and is therefore included in the US model as an additional endogenous variable. Due to a closed system, the real exchange rate does not feature in the US model and $e^*_{0t} - p^*_{0t}$ (foreign price level converted into US$) is included in the US model as a foreign variable.

[17] If additional countries/regions, for example the Euro area, were treated as being locally dominant over some of their neighbours, then the number of parameters to be estimated per equation would be even higher.

weak exogeneity assumption was found to be rejected across various lag specifications. This is consistent with the hypothesis that the dominance of the US is likely to occur predominantly via financial markets. Similar findings are reported in DdPS. As in DdPS, foreign variables that did not pass the weak exogeneity tests were excluded from the US model.

The second constructed GVAR, also referred to as the extended GVAR model (EM), follows the suggestions of Section 3.2 and assumes that the US is a dominant economy. In this case, the restrictions imposed in the BM need not be in line with the true international linkages that exist between economies, even asymptotically for large number of countries. See Chudik and Straub (2011) for a similar argument in the context of the asymptotic analysis of equilibrium in a theoretical N-country DSGE model. Therefore the vector of foreign variables in the individual country models also includes US variables

$$\left(y_{it}^{*}, \pi_{it}^{*}, \rho_{it}^{s*}, p_t^o, \pi_{0t}, y_{0t}, \rho_{0t}^{s}\right), \text{ for } i > 0 \text{ (EM only)}. \tag{66}$$

The main difference in relation to the BM is thus the relaxation of the restrictions given by the specification of individual country models. Note that the analysis of the IVAR model in Section 3.2 implies that US variables and US-specific foreign star variables should, in general, be treated as endogenous. This suggestion is considered in the Supplement.[18] Estimating an unrestricted VAR in 8 endogenous variables is, given the availability of data, not very reliable. The US country models in the BM and EM explored below are chosen to be the same.

5.1 Aggregation Weights

As in DdPS and Dées, Holly, Pesaran, and Smith (2007), a fixed foreign trade matrix (based on the average trade flows computed over the years 1999-2001) was used to construct country-specific foreign variables. The full foreign trade matrix is given in DdPS (2007, Supplement B). Besides the size of individual economies, the largest column sums of the trade matrix plotted in Figure 1, provide useful pieces of additional information on the position of the economy in the panel. The US is the first, followed very closely by the Euro area. The drawback of Figure 1 is that it contains information only on the trade of goods. Yet the dominance of the US is likely to occur predominantly via the financial markets.

5.2 Unit Root Tests

The order of integration of the individual series under consideration were initially examined. Assuming variables are $I(1)$ allows long-run relationships to be interpreted as cointegrating. Real output, inflation and interest rates are commonly found to be $I(1)$ in the literature. Supplement A of DdPS reports WS-ADF unit root tests of Fuller and Park (1995) for all variables in the panel. Table 9 in the Appendix extends the unit root tests presented in DdPS by adopting the modified

[18] After consistent estimation of the US model featuring 8 endogenous variables, $\mathbf{x}_{0t} = (\pi_{0t}, y_{0t}, \rho_{0t}^{s}, p_t^{o})'$ and $(y_{0t}^{*}, \pi_{0t}^{*}, \rho_{0t}^{s*}, e_{0t}^{*} - p_{0t}^{*})'$, a conditional model for $\mathbf{x}_{0t}$ (conditioning is on foreign variables) is derived. This derived conditional model is used in the supplement for the construction of a new version of a GVAR.

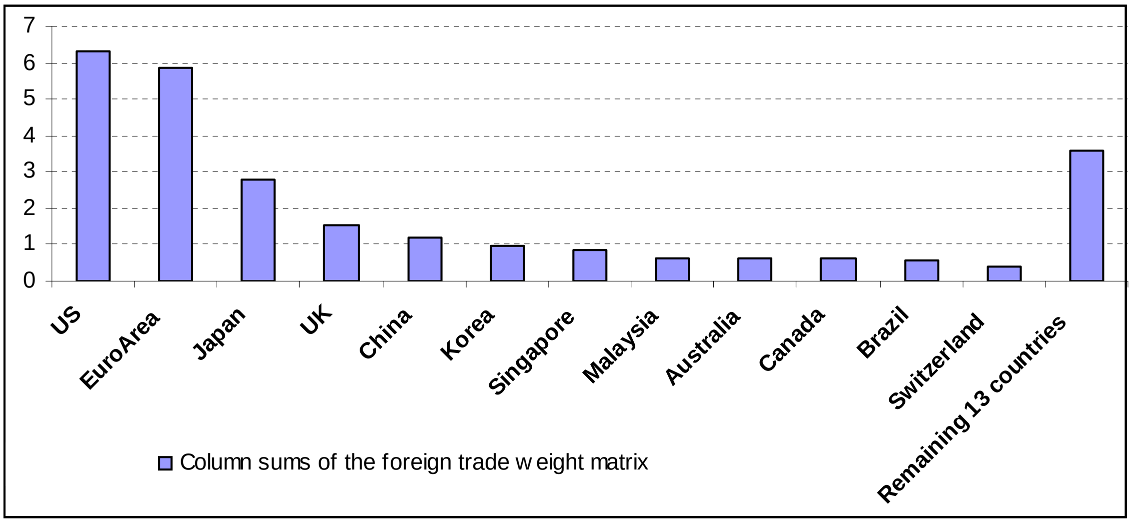

Figure 1: The largest column sums of the foreign trade share matrix constructed from IMF DOTS database for 26 countries/regions of the panel.

AIC criterion proposed by Ng and Perron (2001), which is found to have better finite sample properties than traditional AIC. DdPS results were largely confirmed, with the UK output no longer being a borderline case.[19]

5.3 Estimation of the Country Models

Assuming foreign variables are weakly exogenous and the parameters are stable over time, individual VARX* models were estimated using techniques developed by Harbo et al. (1998) and Pesaran, Shin, and Smith (2000).[20] Both of these assumptions are tested in the following subsections. Lags for the endogenous variables in the individual VARX* models, denoted by p_i, were chosen by AIC. Owing to the data constraints, p_i was not allowed to be greater than 2 and lags for the weakly exogenous variables, denoted by $q_i = r_{0i} = r_{1i}$, were set as equal to 1. Cointegration tests were subsequently conducted. The rank of the cointegrating space was determined according to trace statistics, which is known to have better finite sample performance than the max eigenvalue statistics. Asymptotic 5% critical values taken from MacKinnon, Haug, and Michelis (1999) were employed in the tests.[21] The deterministics considered were unrestricted intercept and restricted trends (also referred to as the 'case IV' in the literature). Individual country models were then estimated subject to the reduced rank restrictions and the corresponding error correcting terms (ECM) were derived. ECMs were subsequently used to conduct weak exogeneity tests.

The number of cointegrating relationships in the BM country VARX* models, which do not include US variables as additional weakly exogenous regressors, and the selected lag length for

[19] Besides the WS ADF unit root tests, the ADF tests with GLS de-trending and the MAX ADF test proposed by Leybourne (1995) with GLS de-trending were also performed. These tests confirmed the findings of the unit root tests in Table 9 and are therefore omitted due to space considerations.

[20] Estimation strategy is the same as in PSW, and DdPS versions of GVAR. See also Footnote 13.

[21] Detailed cointegration tests are presented in the Supplement due to space considerations.

the domestic and foreign variables are reported in Table 2. Table 6 presents the same results for the EM country models. For comparison, the tables also report the number of cointegrating relationships in the corresponding VAR models that contain only domestic endogenous variables plus the real price of oil (p_t^0). The preferred specification seems to be VARX* $(2, 1)$. There were 49 cointegrating relationships in the BM compared to 52 in the EM. These cointegrating relationships will be assessed by means of persistence profiles in the constructed GVAR models in Subsection 5.7 below.

5.4 Weak Exogeneity Tests

Weak exogeneity of foreign variables with respect to the long-run parameters of the conditional model was one of the assumptions underlying the estimation strategy. After estimating the individual VARX* models, weak exogeneity was formally tested for along the lines of Harbo et al. (1998) and Johansen (1992). In particular, F-tests for the joint significance of the error-correcting terms taken from the individual (conditional) VARX* models in the partial models for the corresponding postulated weakly exogenous variables were conducted. The number of lags for the variables in the partial models were the same as those in the corresponding conditional models in the reported test results in Table 1 for BM and Table 5 for EM. Also investigated was the sensitivity of these tests to the different choices of lags. Very similar results were found to those presented in the Appendix.

Weak exogeneity assumptions were rejected at the 5% level in the BM (Table 1) in 3 out of 102 cases, representing a fraction of only 3%. Tests results reported in Table 5 confirm the weak exogeneity of foreign variables in the EM. Weak exogeneity was rejected at the 5% level in only 4% of cases (7 out of 177 tests). Among the foreign US variables, weak exogeneity was rejected in the case of US inflation in Malaysia and US output in Japan (p-value is 3.6%) and Chile. Three rejections (at the 5% significance level) out of 75 tests again represent a fraction of only 4%.

5.5 Structural Stability Tests

The same battery of structural stability tests as performed in DdPS were conducted. Although, in the context of cointegrated models, the possibility of a structural break is relevant for both long-run as well as short-run coefficients, the focus was on the stability of short-run coefficients, as the availability of data hinders any meaningful tests of the stability of cointegrating vectors.

The following tests were performed; Ploberger and Krämer (1992) maximal OLS cumulative sum (CUSUM) statistics, denoted by PK_{sup}; its mean square variant, denoted by PK_{msq}; Nyblom's (1989) tests for the parameter constancy against non-stationary alternatives, denoted by $\mathfrak{N}$; the Wald form of Quandt's (1960) likelihood ratio statistics, denoted by QLR; the mean Wald statistics of Hansen, denoted by MW; and Andrews and Ploberger (1994) Wald statistics based on exponential average, denoted by APW. The last three tests are Wald type tests utilizing a single break at an unknown point. The heterokedasticity-robust version of the tests were also conducted. Stability tests performed were based on residuals of the individual country models, which depend on the dimension of the cointegrating space, not on the identification of cointegrating relationships. The results of these tests at 5% significance level are reported while critical values of the tests, computed under the null of parameter stability, were calculated using the sieve bootstrap samples. Details of the bootstrap procedure is given in DdPS (2007, Supplement A).

Table 4 summarizes the stability tests for BM. The findings were similar to the DdPS version of GVAR. Test results varied across the tests and to a lesser extent across variables. In particular, among $\mathfrak{N}$, QLR, MW and APW tests, outcomes depended to a large extent on whether heteroskedasticity-robust versions of the tests were employed. Non-robust versions showed relatively large rejections, while coefficients seemed reasonably stable using robust versions of the tests. PK tests (both versions) rejected the null of parameter constancy in only 6.8% of cases. As in DdPS, the main reason for the rejection seems to be breaks in the error variances as opposed to parameter coefficients. Once breaks in error variances are allowed for, parameters seems to be reasonably stable. Turning to the tests outcomes summarized in Table 8 for EM, the pattern of rejections was similar to BM. There were fewer numbers of overall rejections across all tests with the exception PK_{sup} (the same number of rejections as in BM) and robust-APW (11 rejections out of 103 tests).

In accordance with DdPS, the problem of possible breaks in error variances was dealt with by using bootstrap means and bootstrap confidence intervals in the persistence profiles and in the generalized impulse responses analysis, and by using robust standard errors for constructing t-ratios for the impact elasticities of the foreign variables given below.

5.6 Contemporaneous Effects (Impact Elasticities) of Foreign Variables on Their Domestic Counterparts

The contemporaneous effects of foreign variables on their domestic counterparts are presented in Table 3 for BM and Table 7 for EM. t-ratios were computed using White's heteroskedasticity consistent variance estimator.

Findings in the BM were similar to the DdPS version of the GVAR. Contemporaneous effects were significant in about 45% of the cases with the average impact elasticity for output being 61% (significant in 10 out of 26 cases), 67% for inflation (significant in 12 out of 26 cases) and 51% for short term interest rates (significant in 9 out of 25 cases).

The EM included US variables as additional weakly exogenous foreign variables in the individual VARX* models. Note that the star variables are weighted averages of all foreign trade partners, thus they themselves include US variables by construction. The significance of additional US variables $(\pi_{0t}, y_{0t}, \rho_{0t}^s)$ in the non-US country models would imply that the relaxation of the restrictions in BM improves the model. Contemporaneous effects of the US foreign variables on their domestic counterparts were found to be significant in 18 out of 74 cases, including the models for the Euro area, Japan and UK. This represent a fraction of 24.3% , well above the nominal 5% level of the tests. Contemporaneous effects of star variables remain significant in about 40% of cases. It is interesting that in the case of Canada additional foreign US variables were not significant. This suggests that either the power of the tests in this case is low (star variables for Canada were highly correlated with US variables simply because US variables have a large weight in the construction of Canadian star variables) or that the impact of the US on Canada successfully comes through the star variables, i.e. that the restriction imposed by the trade weights in construction of star variables is valid. On average, the impact elasticity of y^*_{it} on domestic output was 74%, π^*_{it} on domestic inflation was 88% and ρ^*_{it} on domestic short term interest rates was 37%. The impact elasticity of the additional US foreign output variable on domestic output was on average negative, -16%, suggesting that the weight of the US output in the construction of star variables is probably overstated on average, at least in the short-run. Note that the negative elasticity here does not imply that US output, on average, has an instantaneously negative impact on the output in remaining countries.[22] Also, in the case of the foreign US short term interest rate variable, the average impact elasticity on domestic short term interest rates was negative, -25%. Finally, in the case of US inflation, the average of impact elasticity on domestic inflation was positive, 27%, suggesting that on average, weights of US inflation in π^*_{it} are probably understated, at least in the short-run. It would be interesting in future research to compare these results with the weights derived from a theoretical N-country DSGE model calibrated to data. Note that the relaxation of the trade weight restrictions achieved by including the US variables allows for both the short-run and long-run impact of the US variables on the domestic economy to be chosen by data.

5.7 Construction of GVAR and Persistence Profiles

There are 26 individual country models and 103 endogenous variables in both models. Individual country models were solved together in one system using steps described in detail by DdPS. The largest eigenvalue of the solved GVAR model was 1 in both the BM and EM versions. There were exactly 54 eigenvalues equal to one in the BM, which corresponds to the number of endogenous variables minus the number of cointegrating relationships ($54 = 103 - 49$). The same was true for the EM, where the number of eigenvalues equal to unity was $51 = 103 - 52$. The largest eigenvalue in absolute value inside the unit circle was 0.928 in the BM and 0.929 in the EM.

Persistence profiles (PP) introduced by Pesaran and Shin (1996) were used to examine the effect of system-wide shocks on the dynamics of the long-run relations. PPs are based on a moving average representation of the GVAR and they refer to the time profile of the effects of a shock on the cointegrating relations. See Dées, Holly, Pesaran, and Smith (2007) for a theoretical exposition of PPs in the context of GVAR. PPs have a value of 1 at the time of impact and should converge to zero as the time horizon reaches infinity. Thus PPs allows us to examine the speed at which the long-run relations converge to their equilibrium states.

Bootstrap means of PPs are plotted in Figure 2 for BM and Figure 3 for EM. After a shock, all variables returned to their long-run equilibrium. In most cases, the convergence was quite rapid, often taking less than 2 years. In 28 out of 49 cointegrating relationships the value of the PPs was less than 20% after one year in the BM and only in the case of UK (for 2 cointegrating relationships), Japan (for 2 cointegrating relationships) and Peru (for 1 cointegrating relationship) was the PP more than one-third after two years. Convergence towards long-run equilibrium was found to be more rapid in the EM, where the value of the PPs was less than 20% after one year in 39 out of 53

[22] The net contemporaneous effect of US output on the output in remaining countries would have to be calculated from impact elasticities of y^*_{it} and y_{0t}, using the corresponding US weight w_{i0} in the construction of y^*_{it}.

cointegrating relationships and no PP exceeded 20% after two years.

5.8 Generalized Impulse-Responses

The dynamic properties of the two developed versions of GVAR were investigated further by means of generalized impulse response functions (GIRF). Compact theoretical expositions of GIRFs within the context of a GVAR model is provided in Dées, Holly, Pesaran, and Smith (2007). Four sets of GIRFs were constructed. A negative unit (1 s.e.) shock to US variables (real output, inflation and interest rates) and to the real price of oil was introduced. The impact on the four largest economies based on average nominal output, the US, the Euro area, Japan, and UK, is presented.

5.8.1 Shock to US Real Output

Negative unit shock to US GDP corresponds to a 0.5% decline in US real output. Corresponding GIRFs are presented in Figure 4. Conditional US country models in the EM and BM are identical; hence it is not surprising that the response of US variables to the shock was similar across BM and EM. This is also true of other shocks to US variables. It is interesting to note that bootstrap confidence intervals were smaller in the EM model for GIRFs characterizing the response of US indicators.

In the case of UK macroeconomic variables, the impact of a half percent decline in US GDP becomes statistically significant in the EM: UK GDP suffered approximately a 0.25% decline in one year, the UK interest rate dropped by 15 basis points (bp) in two years, while UK inflation was only marginally affected (less than a 0.1% decline). Macroeconomic indicators in Japan, on the other hand, showed no significant responses to US GDP shock in the EM model, except from a very small 3bp instantaneous increase in interest rates. This is despite considerably smaller bootstrap confidence bounds than the BM. The BM on the other hand shows a significant long-run impact on inflation and interest rates in Japan.

The responses of Euro area indicators were similar across the two models. Output was affected only marginally, inflation showed no significant response and interest rates declined by about 6bp over a longer period.

5.8.2 Shock to US Inflation

Figure 5 plots the GIRFs of a negative unit shock to US Inflation, which corresponds to a 0.38% decrease. The shock induced an increase in US GDP (about 0.6%-0.8% in two years) and a slight 6bp drop in US interest rates.

With the exception of Japan, foreign real output was positively affected. A circa 0.5% increase in Euro area output in 2 years and a 0.3% rise in UK output in one year was noted. Foreign inflation in general declined. In the case of the Euro area and Japan, the response of inflation to the shock was more pronounced in the EM model with much smaller bootstrap bounds. The response of UK inflation was similar across the two models.

Turning to interest rates, there was a significant drop in Japanese short term interest rates, around 8bp in 1 year. Interest rates in the Euro area showed no statistically significant response in the BM and a marginally significant response in the EM a few quarters after the shock. The two responses have opposite signs. The response of UK interest rates was significantly positive in the BM (10bp increase in 2 years) and statistically insignificant in the EM.

5.8.3 Shock to US Short-Term Interest Rates

Turning to Figure 6 which plots the GIRFs of a negative unit shock to US interest rates, we see that the price puzzle (Sims, 1992; Eichenbaum, 1992), also observed in the DdPS version of GVAR, remains. In particular, a 20 bp decrease in US short term interest rates was accommodated by a decrease in inflation. There was also a slight drop in US output, although this response became positive in about 1-2 years, but was statistically insignificant.

The biggest difference between the EM and BM GIRFs can be seen in the case of the Euro area and UK interest rates. Here the responses were more pronounced in the EM. Euro area interest rates dropped by 12bp in one year. Similar pattern were observed in the UK interest rates response.

5.8.4 Shock to the Real Price of Oil

The GIRFs of a negative unit shock to the real price of oil in Figure 7 were similar across the EM and BM with the exception of interest rates in Japan and inflation in the UK and Japan. Interest rates in Japan were less affected in the EM, showing about a 5bp decline in one year compared to 12bp in the BM. Inflation in the UK and Japan, on the other hand, exhibited more oscillatory behavior in the EM.

The shock represented about a 12% instantaneous decline in the real price of oil. Impact on inflation across countries was in general negative, which is intuitive. Inflation dropped by 0.2%-1.2% after one year, depending on the country. The response of real output to the shock was significant by a margin only in the case of the UK while statistically insignificant in the US, Euro area and Japan. With the exception of Japan (and marginally in the Euro area in the BM) interest rates were left unaffected.

Overall, in 40 of the 48 GIRFs in Figures 5-7 bootstrap confidence bounds are found to be smaller in the EM when compared to the BM.

6 Concluding Remarks

In this paper, macroeconometric modelling from a global perspective, treating the US economy as globally dominant, has been considered. Following Chudik and Pesaran (2011 and 2012), the GVAR approach is derived as an approximation to an infinite dimensional VAR model featuring variables integrated of order one. It has been highlighted that the restrictions commonly imposed by researchers to overcome the curse of dimensionality in open economy macroeconomics are mis-specified in the presence of a globally dominant economy (assumed to be the US). In this case, individual country models for small open economies need to be augmented by US variables in addition to weighted cross sectional averages of foreign trading partners (star variables).

This suggestion was followed in the estimation of a GVAR model featuring 33 countries during the period 1979(Q2)-2003(Q4). Two GVAR models were constructed, each featuring the same set of variables: real output, interest rates, inflation, the real price of oil and exchange rate variables. The first is the benchmark GVAR model (BM), which is similar to the Dées et al. (DdPS, 2007) model. The second, the extended GVAR model (EM), relaxed the restrictions imposed in the BM. Individual country models in the EM were augmented by both US and star variables in line with the US being treated as globally dominant. Evidence in support of the EM is presented and the two models are compared by means of persistence profiles of system wide shocks to the cointegrating relationships and by means of generalized impulse response functions. The choice of whether the US economy should be treated as the "globally dominant" economy in the GVAR is shown to be of importance when assessing the magnitude of international economic linkages.

While statistical evidence was found for relaxing the restrictions assumed in the BM, treating the US as globally dominant has its cost in terms of degrees of freedom. Problem of finding balance between complexity of estimated model and availability of data has not been addressed. A promising avenue for future research would be to combine limiting restrictions considered in this paper with Bayesian restrictions in the empirical work. A closer integration of theoretical DSGE models and GVAR remains also an interesting and unexplored topic.

Table 1: Weak exogeneity tests of the country-specific foreign variables in the BM.

Country	F-test	y_{it}^{*}	π_{it}^{*}	ρ_{it}^{s*}	p_t^{o}
China	$F(1,76)$	0.533	1.055	0.718	1.074
Euro Area	$F(2,79)$	1.732	0.589	0.856	0.267
Japan	$F(3,78)$	2.115	**3.763**	1.171	1.076
Argentina	$F(2,75)$	2.203	0.516	2.409	2.232
Brazil	$F(1,76)$	0.040	1.266	0.314	0.379
Chile	$F(2,75)$	3.105	0.533	0.554	0.329
Mexico	$F(3,78)$	2.072	0.526	0.075	0.490
Peru	$F(3,74)$	0.196	0.558	0.682	2.400
Australia	$F(1,76)$	0.566	0.003	2.262	0.197
Canada	$F(3,78)$	0.638	0.556	1.769	1.020
New Zealand	$F(1,76)$	0.038	0.973	0.339	1.931
Indonesia	$F(3,74)$	0.329	1.454	0.353	0.477
Korea	$F(3,74)$	2.026	0.586	0.533	1.170
Malaysia	$F(1,76)$	0.147	0.952	0.019	1.021
Philippines	$F(2,75)$	0.535	1.470	0.249	**3.793**
Singapore	$F(1,80)$	0.501	1.913	0.434	0.233
Thailand	$F(2,79)$	2.974	0.093	2.012	2.182
India	$F(2,75)$	0.009	2.113	1.261	1.900
South Africa	$F(1,76)$	0.627	0.425	0.819	0.057
Saudi Arabia	$F(1,78)$	0.005	2.124	0.014	0.032
Turkey	$F(1,76)$	1.969	0.801	0.430	0.091
Norway	$F(2,75)$	0.034	0.665	0.342	1.03
Sweden	$F(2,75)$	1.601	0.312	0.052	0.622
Switzerland	$F(2,79)$	3.101	0.250	0.475	0.928
UK	$F(3,74)$	**2.836**	0.924	0.414	0.819
US	$F(1,80)$	0.221	0.058	...	...

Note: Significant coefficients at 5% level are highlighted.

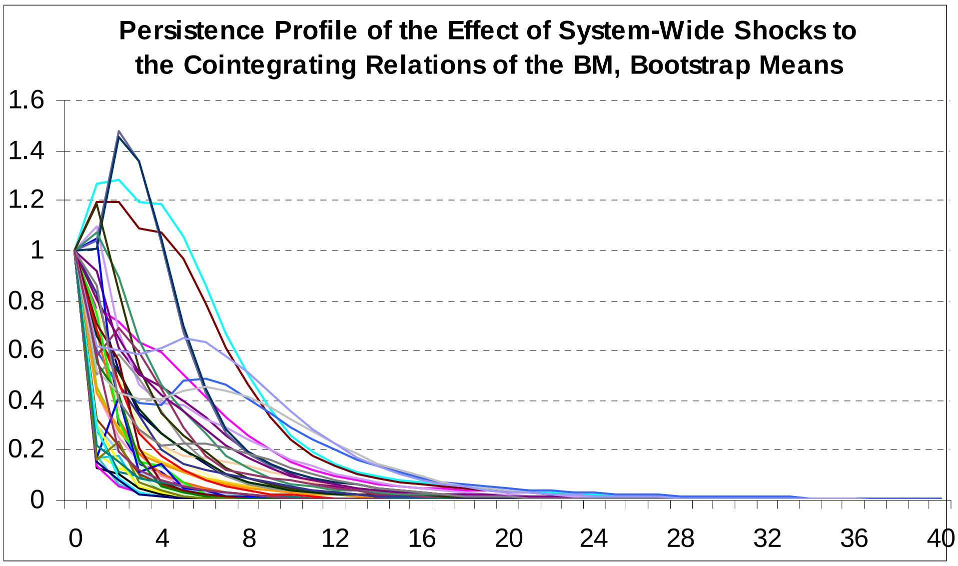

Figure 2: Persistence Profile of the Effect of System-Wide Shocks to the Cointegrating Relations of the Benchmark GVAR Model, Bootstrap Means

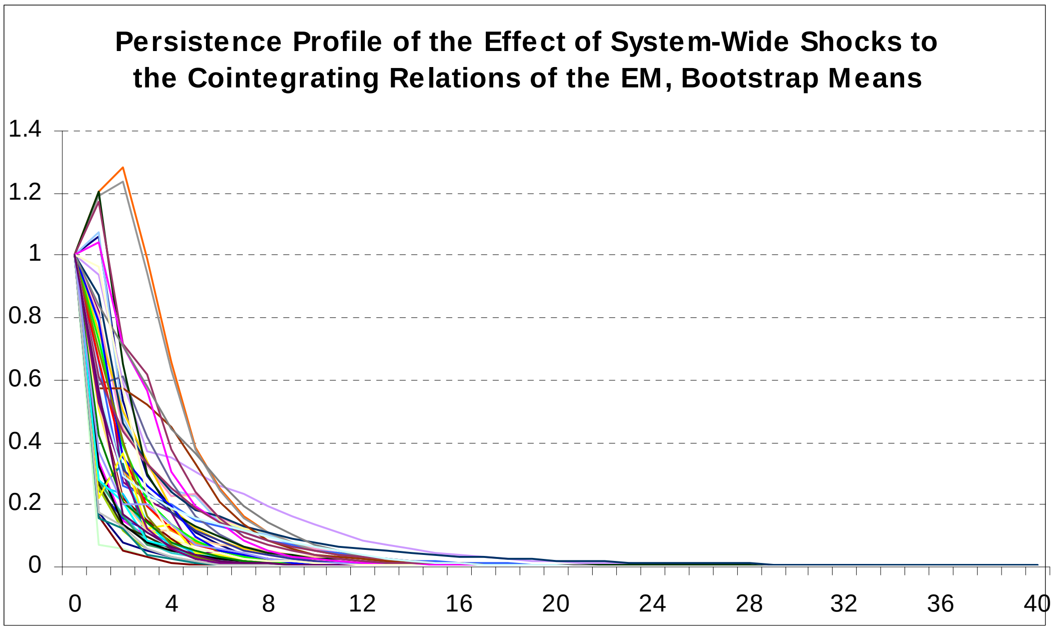

Figure 3: Persistence Profile of the Effect of System-Wide Shocks to the Cointegrating Relations of the Extended GVAR Model

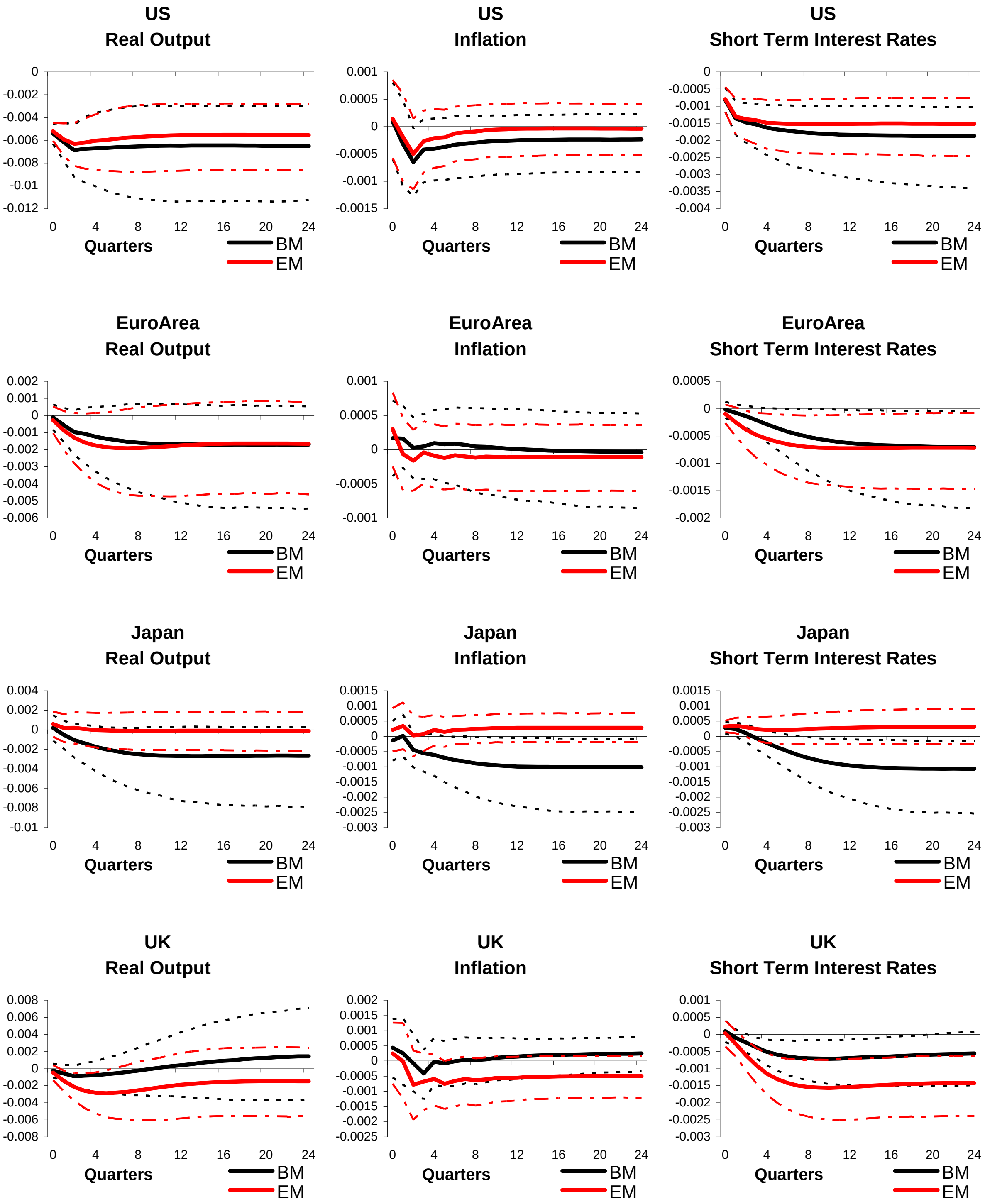

Figure 4: Generalized impulse responses of a negative unit (1 s.e.) shock to US real output. Bootstrap estimates with 90% bootstrap error bounds.

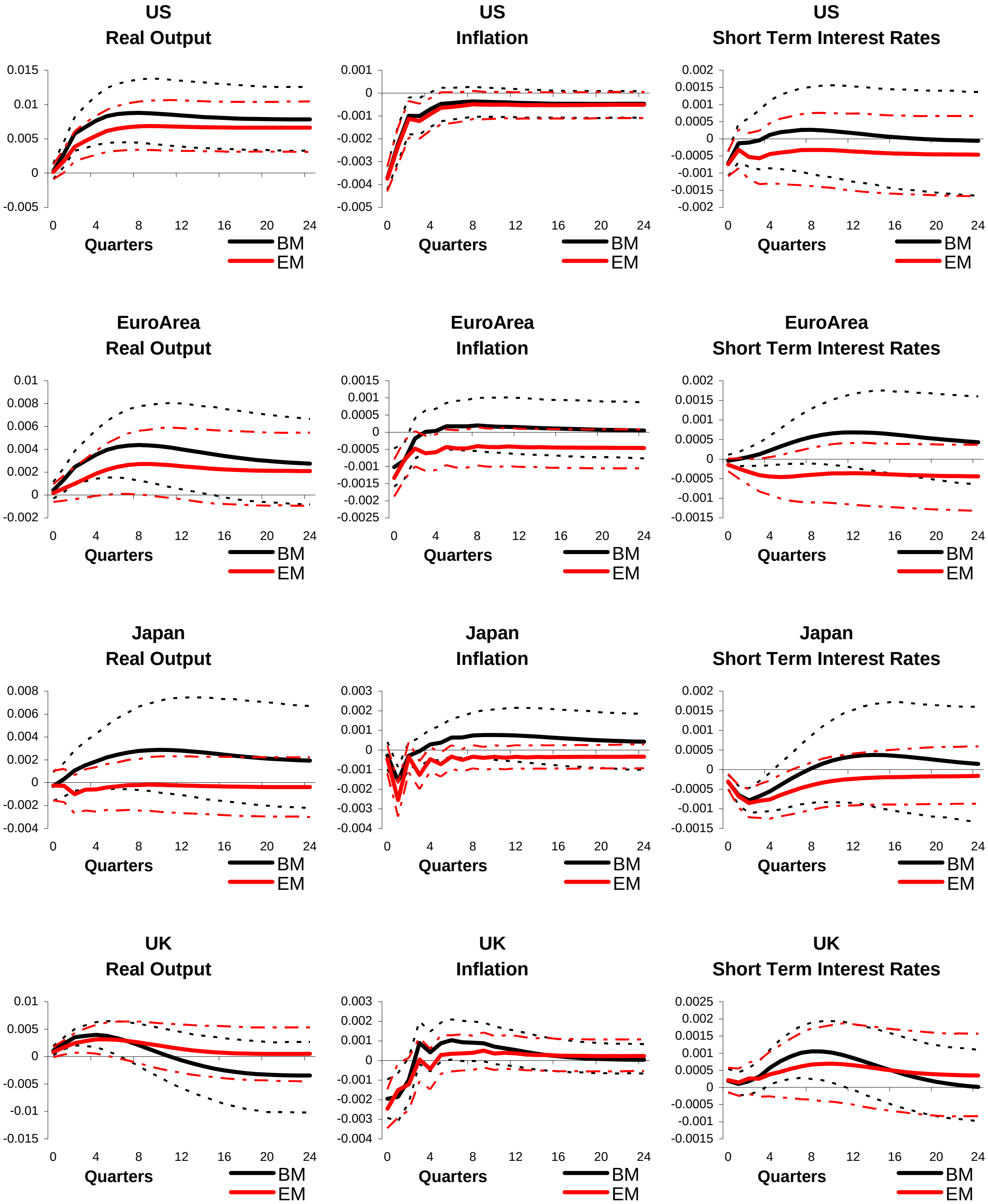

Figure 5: Generalized impulse responses of a negative unit (1 s.e.) shock to US inflation. Bootstrap estimates with 90% bootstrap error bounds.

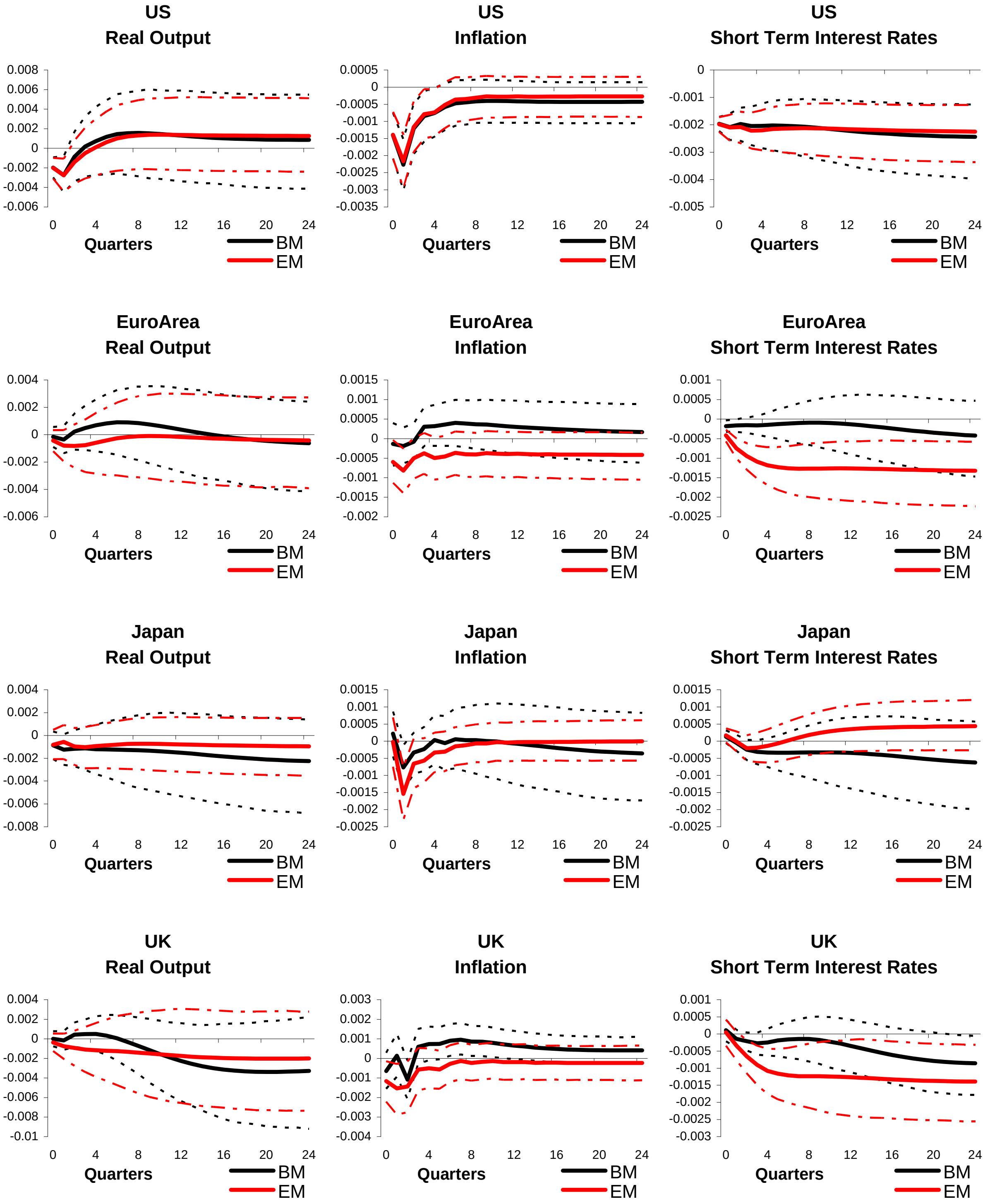

Figure 6: Generalized impulse responses of a negative unit (1 s.e.) shock to US short term interest rates. Bootstrap estimates with 90% bootstrap error bounds.

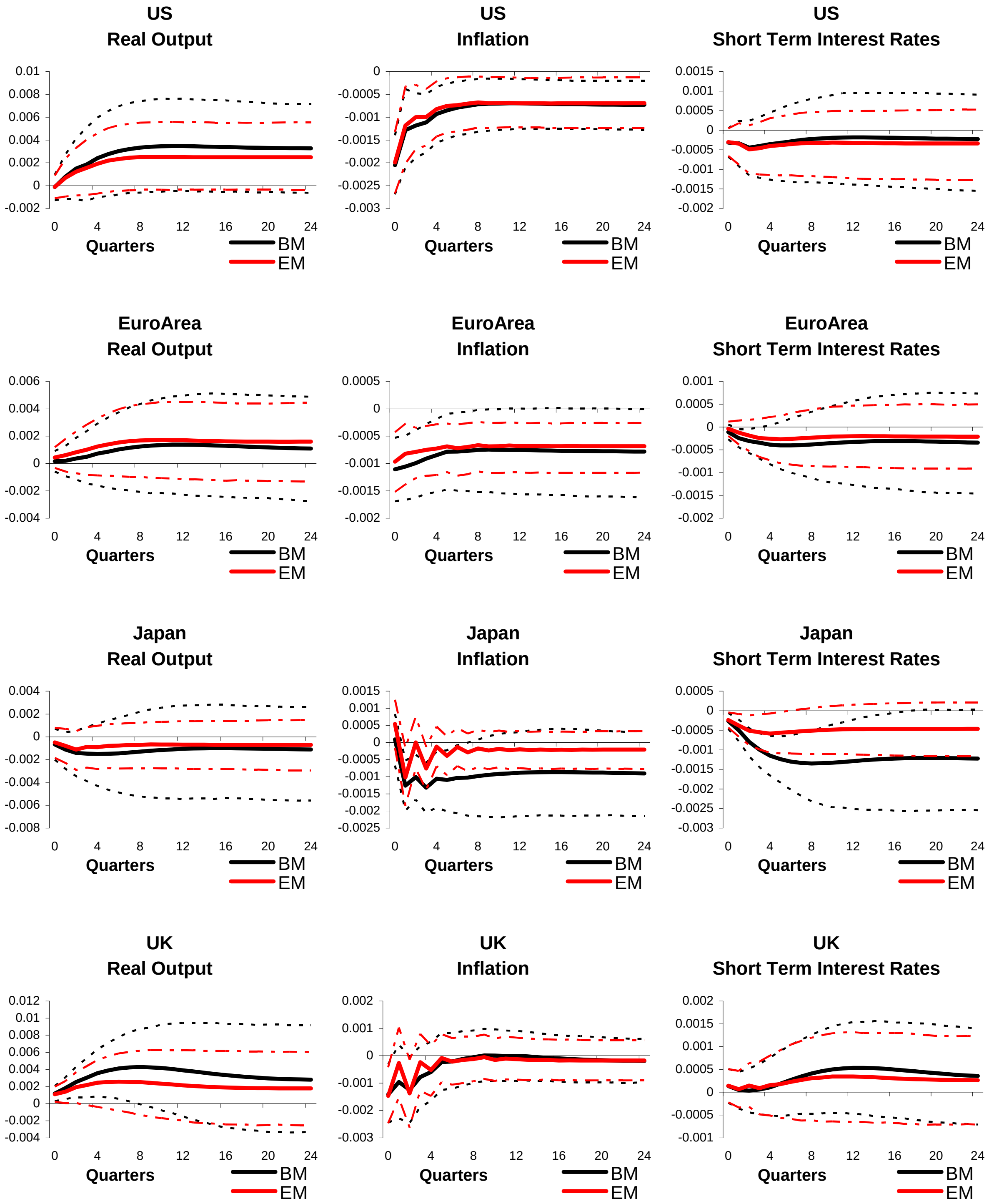

Figure 7: Generalized impulse responses of a negative unit (1 s.e.) shock to real price of oil. Bootstrap estimates with 90% bootstrap error bounds.

Table 2: Number of cointegrating relationships and lags in the BM.

Country	No. of cointegrating relationships VARX models	VAR models	p_i (based on AIC)	q_i
China	1	1	2	1
Euro Area	2	2	1	1
Japan	3	1	1	1
Argentina	2	1	2	1
Brazil	1	1	2	1
Chile	2	2	2	1
Mexico	3	1	1	1
Peru	3	2	2	1
Australia	1	1	2	1
Canada	3	2	1	1
New Zealand	1	1	2	1
Indonesia	3	2	2	1
Korea	3	2	2	1
Malaysia	1	1	2	1
Philippines	2	2	2	1
Singapore	1	1	1	1
Thailand	2	2	1	1
India	2	2	2	1
South Africa	1	1	2	1
Saudi Arabia	1	2	2	1
Turkey	1	1	2	1
Norway	2	1	2	1
Sweden	2	1	2	1
Switzerland	2	1	1	1
UK	3	2	2	1
US	1	1	2	1
All countries	49	37		

Notes: In the case of the individual VAR models (without star variables) price of oil is included as an endogenous variable in all models. Number of cointegrating relationships is selected according to Mackinnon's 5% asymptotic critical values.

Table 3: Contemporaneous effects of foreign variables on their domestic counterparts in the benchmark model.

	Star variables		
	y_{it}^{s}	π_{it}^{*}	ρ_{it}^{s}
China	−0.06 (−0.54)	**0.61** (2.44)	0.10 (1.74)
Euro Area	**0.62** (6.75)	**0.30** (5.25)	**0.09** (4.32)
Japan	**0.35** (2.06)	−0.02 (−0.21)	−0.08 (−1.30)
Argentina	0.33 (1.33)	−3.12 (−1.56)	**3.80** (2.14)
Brazil	**0.77** (2.46)	1.10 (1.48)	0.98 (0.67)
Chile	0.41 (1.20)	0.03 (0.77)	**0.06** (2.36)
Mexico	**0.83** (3.51)	1.02 (1.67)	−0.20 (−0.32)
Peru	0.52 (0.84)	1.57 (0.58)	−0.51 (−0.37)
Australia	**0.51** (2.67)	**0.37** (2.36)	**0.39** (2.04)
Canada	**0.54** (5.05)	**0.56** (4.53)	**0.48** (2.74)
New Zealand	**0.61** (2.52)	0.15 (0.60)	0.31 (1.09)
Indonesia	0.73 (1.90)	−0.17 (−0.30)	1.12 (1.91)
Korea	0.80 (1.99)	**1.72** (4.87)	−0.17 (−1.24)
Malaysia	**1.42** (5.16)	**0.46** (2.05)	0.07 (0.39)
Philippines	0.07 (0.19)	−0.83 (−1.87)	**0.93** (2.48)
Singapore	**1.41** (7.25)	**0.37** (2.35)	**0.49** (2.54)
Thailand	0.20 (0.74)	**1.14** (4.44)	**1.01** (2.45)
India	0.24 (0.82)	−0.57 (−1.23)	−0.29 (−1.05)
South Africa	0.25 (1.53)	0.04 (0.15)	0.07 (0.64)
Saudi Arabia	0.61 (1.49)	−0.02 (−0.09)	...
Turkey	0.56 (0.77)	**9.59** (8.35)	1.84 (1.77)
Norway	**0.88** (2.27)	**0.85** (5.72)	0.27 (0.98)
Sweden	**1.26** (3.44)	**1.38** (7.10)	**1.18** (3.06)
Switzerland	**0.73** (5.65)	**0.44** (3.32)	0.12 (1.30)
UK	**0.50** (3.68)	0.38 (1.62)	0.19 (1.25)
US	**0.75** (5.12)	0.11 (1.44)	...

Note: Significant coefficients at 5% level are highlighted.

Table 4: Stability tests, BM

Test Statistics	y_{it}	π_{it}	$p_{it}-e_{it}$	ρ^s_{it}	Total(%)
$\mathrm{PK}_{\mathrm{sup}}$	2(7.7)	2(7.7)	2(8.0)	1(4.0)	7(6.8)
PK_{msq}	2(7.7)	2(7.7)	2(8.0)	1(4.0)	7(6.8)
N	0(0.0)	5(19.2)	4(16.0)	6(24.0)	15(14.6)
robust-N	1(3.8)	3(11.5)	(21.0)	3(12.0)	10(9.7)
QLR	7(26.9)	12(46.2)	9(36.0)	15(60.0)	43(41.7)
robust-QLR	3(11.5)	2(7.7)	3(12.0)	3(12.0)	11(10.7)
MW	5(19.2)	9(34.6)	7(280)	9(36.0)	30(29.1)
robust-MW	3(11.5)	3(11.5)	2(8.0)	4(16.0)	12(11.7)
APW	8(30.8)	11(42.3)	8(32.0)	15(60.0)	42(40.8)
robust-APW	3(11.5)	1(3.8)	1(4.0)	3(12.0)	8(7.8)

Table 5: Weak exogeneity tests of the country-specific foreign variables in the EM.

Country	F-test	y^*_{it}	π^*_{it}	ρ^{s*}_{it}	p^o_t	y_{0t}	π_{0t}	ρ^s_{0t}
China	$F(1,70)$	0.183	0.734	0.303	0.082	1.588	2.517	2.255
Euro Area	$F(1,70)$	1.297	0.028	0.037	0.009	0.002	0.001	2.032
Japan	$F(2,73)$	**6.158**	0.91	1.560	0.765	**3.479**	1.046	0.126
Argentina	$F(1,70)$	1.882	0.660	3.508	2.795	0.066	2.055	0.017
Brazil	$F(2,69)$	0.432	1.725	0.466	0.050	0.399	0.350	1.265
Chile	$F(3,68)$	1.841	0.813	0.431	0.523	**4.275**	0.322	0.314
Mexico	$F(1,74)$	0.426	0.378	0.529	0.996	0.602	1.000	0.087
Peru	$F(3,78)$	0.654	0.597	0.824	0.980	0.451	0.592	0.250
Australia	$F(3,72)$	0.880	1.647	0.428	**2.804**	0.090	0.502	1.066
Canada	$F(3,72)$	0.633	0.898	1.077	1.943	0.707	1.222	0.722
New Zealand	$F(2,69)$	1.114	2.832	0.798	1.260	1.334	2.128	0.432
Indonesia	$F(3,68)$	0.625	1.470	0.268	0.688	0.066	1.694	1.325
Korea	$F(3,68)$	1.024	0.567	1.324	1.211	0.801	0.745	2.717
Malaysia	$F(1,70)$	0.336	0.508	0.104	0.412	0.869	**4.066**	1.612
Philippines	$F(3,68)$	1.648	1.139	0.099	**3.752**	1.104	2.163	0.342
Singapore	$F(2,73)$	0.802	0.959	1.260	0.748	1.707	1.077	0.273
Thailand	$F(2,73)$	4.213	0.222	0.961	1.128	2.899	0.77	0.004
India	$F(2,69)$	0.418	0.454	0.724	2.305	0.633	0.311	0.179
South Africa	$F(2,69)$	0.810	0.615	0.219	0.303	0.264	1.687	0.408
Saudi Arabia	$F(1,72)$	0.000	0.221	0.156	0.024	0.258	0.032	0.501
Turkey	$F(2,73)$	0.031	0.084	2.060	0.089	0.477	0.019	0.185
Norway	$F(2,69)$	0.404	0.684	0.150	1.336	0.714	0.701	1.993
Sweden	$F(2,69)$	2.077	0.657	0.212	1.223	1.153	0.370	0.209
Switzerland	$F(2,69)$	0.685	0.108	0.805	0.638	1.674	0.106	0.345
UK	$F(3,68)$	0.679	0.143	0.358	0.553	1.686	0.601	1.835
US	$F(2,77)$	0.858	0.179	...	...	...	...	...

Note: Significant coefficients at 5% level are highlighted.

Table 6: Number of cointegrating relationships and lags in the EM.

Country	No. of cointegrating relationships VARX models	VAR models	p_i (based on AIC)	q_i
China	1	1	2	1
Euro Area	1	2	2	1
Japan	2	1	1	1
Argentina	1	1	2	1
Brazil	2	1	2	1
Chile	3	2	2	1
Mexico	1	1	1	1
Peru	3	2	2	1
Australia	3	1	1	1
Canada	2	2	1	1
New Zealand	2	1	2	1
Indonesia	3	2	2	1
Korea	3	2	2	1
Malaysia	1	1	2	1
Philippines	3	2	2	1
Singapore	2	1	1	1
Thailand	2	2	1	1
India	2	2	2	1
South Africa	2	1	2	1
Saudi Arabia	1	2	2	1
Turkey	2	1	1	1
Norway	2	1	2	1
Sweden	2	1	2	1
Switzerland	2	1	2	1
UK	3	2	2	1
US	1	1	2	1
All countries	52	37		

Notes: In the case of the individual VAR models (without star variables) price of oil is included as an endogenous variable in all models. Number of cointegrating relationships is selected according to Mackinnon's 5% asymptotic critical values.

Table 7: Contemporaneous effects of foreign variables on their domestic counterparts in the extended GVAR model.

	Star variables			**US variables**		
	y_{it}^{s}	π_{it}^{*}	ρ_{it}^{s}	y_{0t}	π_{0t}	ρ_{0t}^{s}
China	−0.12 (−0.76)	**0.69** (2.20)	**0.13** (2.35)	0.06 (0.45)	0.09 (0.37)	0.03 (0.79)
Euro Area	**0.63** (3.33)	**0.24** (4.75)	**0.05** (2.61)	−0.09 (−0.85)	**0.14** (2.03)	**0.23** (3.83)
Japan	0.42 (1.65)	**−0.39** (−3.21)	0.03 (0.51)	−0.20 (−1.42)	**0.55** (4.69)	**−0.13** (−2.22)
Argentina	0.25 (1.04)	−1.02 (−0.58)	2.90 (1.90)	0.04 (0.14)	5.60 (1.43)	5.24 (1.32)
Brazil	0.61 (1.33)	1.32 (1.54)	0.37 (0.27)	−0.54 (−1.32)	3.02 (1.30)	−12.35 (−1.70)
Chile	0.22 (0.45)	−0.01 (−0.11)	**0.12** (3.23)	−0.17 (−0.47)	−0.28 (−0.49)	−0.98 (−1.02)
Mexico	1.35 (0.57)	**11.30** (3.49)	0.59 (0.50)	−0.69 (−0.33)	**−9.80** (−3.09)	−0.39 (−0.40)
Peru	1.14 (0.84)	1.53 (0.48)	−2.40 (−1.92)	−0.84 (−1.03)	0.57 (0.23)	1.55 (0.88)
Australia	0.27 (1.73)	**0.73** (3.55)	0.33 (1.67)	**0.29** (2.62)	−0.30 (−1.55)	0.06 (0.36)
Canada	0.50 (0.68)	−0.02 (−0.04)	0.54 (1.52)	0.06 (0.08)	0.37 (0.92)	0.11 (0.32)
New Zealand	**0.84** (2.7)	0.47 (1.54)	0.46 (1.45)	−0.34 (−1.76)	−0.36 (−1.39)	−0.24 (−1.12)
Indonesia	**0.87** (2.09)	−0.32 (−0.51)	0.47 (1.06)	−0.08 (−0.24)	0.72 (1.42)	0.73 (1.97)
Korea	**2.64** (3.45)	0.4 (1.14)	**−0.29** (−2.43)	**−0.80** (−1.99)	**1.49** (5.66)	0.27 (1.56)
Malaysia	**1.86** (4.87)	**0.64** (2.22)	0.29 (1.80)	**−0.57** (−2.55)	−0.09 (−0.50)	**−0.31** (−2.25)
Philippines	0.61 (1.56)	**−3.03** (−4.67)	**1.80** (3.55)	**−0.53** (−2.26)	**2.44** (4.33)	**−0.58** (−2.21)
Singapore	**1.49** (7.61)	**0.50** (2.80)	−0.07 (−0.47)	−0.27 (−1.38)	−0.04 (−0.29)	**0.47** (6.05)
Thailand	0.39 (0.96)	**0.72** (2.47)	0.71 (1.59)	−0.19 (−1.06)	0.30 (1.30)	**0.45** (234)
India	0.52 (1.74)	**−1.64** (−3.47)	−0.50 (−1.87)	0.16 (0.91)	**1.55** (4.89)	0.35 (1.07)
South Africa	0.04 (0.18)	**−0.90** (−3.18)	0.14 (1.42)	0.22 (1.75)	**0.68** (3.16)	0.20 (0.81)
Saudi Arabia	0.45 (0.86)	0.24 (0.91)	...	0.24 (0.64)	−0.19 (−0.95)	...
Turkey	0.42 (0.60)	**8.84** (5.80)	1.57 (1.90)	0.34 (0.76)	−0.54 (−0.52)	−0.02 (−0.04)
Norway	0.91 (1.95)	**0.92** (5.72)	0.25 (0.94)	−0.11 (−0.40)	0.03 (0.13)	−0.26 (−1.67)
Sweden	**1.17** (3.55)	**1.20** (5.68)	**1.19** (2.81)	0.03 (0.09)	0.12 (0.81)	−0.29 (−1.13)
Switzerland	**0.74** (3.64)	**0.51** (3.75)	0.06 (0.72)	0.08 (1.02)	0.01 (0.06)	−0.10 (−1.16)
UK	**0.58** (2.88)	−0.26 (−1.15)	0.19 (1.21)	0.76 (4.85)	**0.76** (4.85)	−0.06 (−0.56)
US	**0.75** (5.12)	0.11 (1.44)	...	...	...	...

Note: Significant coefficients at 5% level are highlighted.

Table 8: Stability tests, EM

Test Statistics	y_{it}	π_{it}	$p_{it} - e_{it}$	ρ_{it}^{s}	Total(%)
$\mathrm{PK}_{\mathrm{sup}}$	1(3.8)	2(77)	3(12.0)	1(4.0)	7(6.8)
PK_{msq}	1(3.8)	1(3.8)	1(4.0)	1(4.0)	4(3.9)
N	1(3.8)	3(11.5)	2(8.0)	5(20.0)	11(10.7)
robust-N	1(3.8)	2(7.7)	2(8.0)	1(4.0)	6(5.8)
QLR	7(26.9)	12(46.2)	9(36.0)	11(44.0)	39(37.9)
robust-QLR	0(0.0)	1(3.8)	4(16.0)	1(4.0)	6(5.8)
MW	2(7.7)	7(26.9)	7(28.0)	7(28.0)	23(22.3)
robust-MW	2(7.7)	2(7.7)	2(8.0)	2(8.0)	8(7.8)
APW	8(30.8)	11(42.3)	9(36.0)	10(40.0)	38(36.9)
robust-APW	2(7.7)	3(11.5)	4(16.0)	2(8.0)	11(10.7)

Table 9: WS unit root test statistics for domestic variables (Based on AIC order selection criterion)

	y_{it}			p_{it}			q_{it}			ρ_{it}^{s}			$e_{it} - p_{it}$		
Country	level	Δ	Δ^2	level	Δ	Δ^2	level	Δ	Δ^2	level	Δ	Δ^2	level	Δ	Δ^2
China	−3.75	−3.34	−9.90	−1.63	−2.58	−5.07	.	.	.	−1.20	−4.04	−15.9	−1.47	−4.53	−15.1
E.A.	−1.92	−3.62	−15.9	−2.01	−0.05	−12.5	−2.42	−4.26	−12.8	−1.26	−5.51	−13.0	−2.09	−4.44	−14.7
Japan	−1.36	−3.46	−16.9	−0.60	−0.55	−29.3	−1.40	−4.33	−13.0	−1.61	−5.59	−11.2	−1.52	−4.08	−14.0
Argen.	−2.01	−4.69	−11.1	−1.57	−2.45	−15.4	−3.53	−10.6	−15.3	−2.16	−15.8	−19.1	−1.95	−6.47	−17.7
Brazil.	−2.53	−7.85	−14.7	−1.45	−2.04	−9.11	.	.	.	−2.19	−10.9	−15.5	−2.23	−9.81	−14.5
Chile	−1.80	−3.78	−17.9	0.51	−1.42	−17.7	−1.82	−5.08	−16.2	−1.09	−15.2	−22.0	−2.26	−3.41	−12.6
Mexico	−2.07	−5.29	−18.8	−1.12	−2.16	−10.6	.	.	.	−1.42	−3.94	−14.2	−2.37	−4.42	−16.7
Peru	−2.09	−8.38	−14.7	−1.61	−2.17	−13.2	.	.	.	−3.02	−3.65	−17.3	−1.44	−8.57	−13.4
Australia	−2.12	−6.60	−12.2	−0.40	−1.78	−17.1	−2.89	−9.36	−16.3	−1.86	−3.90	−14.0	−2.11	−3.93	−15.5
Canada	−2.47	−3.86	−12.2	−0.63	−1.31	−13.4	−2.49	−5.73	−15.6	−1.36	−4.19	−21.5	−1.91	−2.19	−14.5
N. Zeal.	−1.21	−4.92	−15.7	0.05	−1.64	−15.2	−2.52	−9.29	−15.5	−1.55	−10.6	−16.8	−2.16	−4.77	−15.6
Indon.	−1.38	−6.04	−17.8	−1.91	−4.90	−7.05	.	.	.	−2.66	−5.34	−19.7	−2.58	−5.79	−8.37
Korea	−1.60	−4.42	−21.0	0.69	−1.20	−13.3	−2.07	−4.54	−14.7	−1.17	−7.91	−12.2	−1.94	−7.43	−12.7
Malays.	−1.84	−4.54	−15.5	−1.33	−2.27	−18.2	−2.42	−4.93	−18.1	−2.41	−7.75	−12.7	−2.27	−6.72	−12.4
Philipp.	−1.17	−2.85	−17.2	−0.85	−3.48	−7.63	−1.36	−5.11	−17.6	−2.31	−8.65	−14.7	−1.91	−5.54	−15.7
Singap.	−1.37	−5.06	−15.8	−0.45	−2.21	−13.7	−3.13	−9.81	−15.9	−1.12	−4.92	−15.0	−0.69	−3.00	−16.6
Thail.	−1.42	−2.57	−13.7	−0.38	−1.35	−15.4	−1.60	−3.46	−14.5	−1.99	−9.57	−15.7	−1.66	−7.29	−14.3
India	−3.15	−7.45	−12.0	−1.01	−3.29	−11.6	−2.39	−7.42	−13.9	−2.64	−16.0	−23.7	−1.70	−7.24	−11.9
S.Africa	−2.39	−4.57	−13.1	0.73	−1.63	−13.0	−3.34	−8.06	−13.5	−1.65	−4.75	−16.6	−1.86	−3.97	−14.4
S.Arabia	−1.35	−4.68	−9.90	−1.82	−3.33	−15.3	.	.	.	.	.	.	−1.62	−3.02	−14.3
Turkey	−2.11	−6.29	−18.0	−1.29	−2.19	−15.9	.	.	.	−0.79	10.2	−16.0	−1.80	−9.26	−14.9
Norway	−2.09	−15.4	−21.8	2.67	−1.38	−15.7	−2.34	−8.02	−13.5	−1.77	−10.7	−17.6	−2.20	−7.70	−13.9
Sweden	−2.83	−14.5	−21.5	−0.77	−1.18	−15.1	−2.50	−4.60	−14.0	−1.96	−14.6	−20.4	−2.00	−3.89	−15.4
Switz.	−1.98	−4.56	−15.0	−1.48	−1.81	−14.1	−1.24	−4.26	−16.4	−1.38	−3.83	−13.3	−2.18	−4.65	−14.2
U.K.	−1.78	−3.21	−16.5	−0.39	−0.61	−19.8	−1.06	−8.46	−14.5	−1.76	−6.37	−18.4	−2.09	−7.90	−14.6
U.S.	−2.23	−4.92	−15.3	−0.11	−0.07	−12.4	−2.07	−5.60	−15.2	−1.10	−3.63	−13.9	.	.	.

WS unit root statistics for price of oil: p_t^o -2.26 Δp_t^o -8.19 $\Delta^2 p_t^o$ -13.15

Notes: The WS statistics for all level variables are based on regressions including a linear trend, except for the interest rate variables. The 95% critical value for the WS statistics for regressions with trend is -3.24, and for regressions without trend -2.55.

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A A More General Nonstationary Infinite-Dimensional VAR(1) Model

Let $p = 1$ and assume that VAR model in $\mathbf{y}_t = \mathbf{x}_t - \boldsymbol{\delta}_0 - \boldsymbol{\delta}_1 t - \boldsymbol{\Gamma}_f \mathbf{f}_t$,

$$\mathbf{y}_t = \boldsymbol{\Phi}\mathbf{y}_{t-1} + \boldsymbol{\zeta}_t, \tag{A.1}$$

satisfies the following assumptions.

ASSUMPTION 7 *Let $\boldsymbol{\Phi}(L) = \mathbf{I}_k - \boldsymbol{\Phi}L$*

(i) The roots of $|\boldsymbol{\Phi}(L)| = 0$ are either outside the unit circle or equal to one.

(ii) The matrix $\boldsymbol{\Pi} = \boldsymbol{\Phi} - \mathbf{I}_k$ has reduced rank $r < k$, i.e. $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $k \times r$ matrices of full column rank.

(iii) The matrix $\boldsymbol{\alpha}'_{\perp}\boldsymbol{\beta}_{\perp}$ has full rank, where $\boldsymbol{\alpha}_{\perp}$ and $\boldsymbol{\beta}_{\perp}$ are orthogonal complements to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

ASSUMPTION 8 *Let $\boldsymbol{\zeta}_t$ be a weakly dependent serially uncorrelated process:*

$$\boldsymbol{\zeta}_t \sim IID\left(\mathbf{0}, \boldsymbol{\Omega}_{\zeta}\right) \tag{A.2}$$

where

$$\varrho\left(\boldsymbol{\Omega}_{\zeta}\right) = O\left(1\right). \tag{A.3}$$

Furthermore, the vector of unobserved common factors $\mathbf{f}_t$ satisfies Assumption 2, and

$$\mathbf{y}_0 \sim \left(\mathbf{0}, \boldsymbol{\Sigma}_0\right), \text{ with } \left\|\boldsymbol{\Sigma}_0\right\|_r = O\left(1\right). \tag{A.4}$$

Starting values $\mathbf{y}_0$ are independently distributed from $\boldsymbol{\zeta}_t$ for any $t \in \{1, 2, ..., T\}$.

ASSUMPTION 9

$$\left\|\boldsymbol{\Phi}\right\|_r = O\left(1\right) \text{ and for any } i \in \mathbb{N} : \left\|\boldsymbol{\Phi}_{-i}\right\|_r = O\left(N^{-1}\right) \tag{A.5}$$

where $\boldsymbol{\Phi}_{-i} = \left(\boldsymbol{\Phi}_{i,0}, ..., \boldsymbol{\Phi}_{i,i-1}, \mathbf{0}_{k_i}, \boldsymbol{\Phi}_{i,i+1}, ..., \boldsymbol{\Phi}_{iN}\right)'$. Furthermore, $\forall \ell \in \mathbb{N}$:

$$\left\|\boldsymbol{\Phi}^{\ell}\boldsymbol{\Phi}'^{\ell}\right\|_r < K < \infty \tag{A.6}$$

Remark 7 *Conditions (i)-(iii) of Assumption 7 are assumptions of the Granger representation theorem.*[23] *Jointly with Assumption 8 and for* finite N*, they ensure the following. Condition (i) ensures that process* $\mathbf{y}_t$ *is not explosive or seasonally cointegrated. The condition (ii) ensures that there are at least* $k-r$ *unit roots and induces cointegration for* $r \geq 1$*. Finally, the last condition, (iii), ensures that the process* $\mathbf{y}_t$ *is not integrated of higher order than* $I(1)$ *as there are exactly* $k-r$ *unit roots.*

Remark 8 *Assumption 9 rules out dominant groups and establishes that the variance of* $\Delta \mathbf{y}_{it}$ *is bounded in* N*. Note that for fixed* N*, the Granger representation theorem implies that* $\lim_{\ell \to \infty} \mathbf{\Phi}^{\ell}$ *exists under Assumption 7, in particular* $\lim_{\ell \to \infty} \mathbf{\Phi}^{\ell} = \boldsymbol{\beta}_{\perp} \left(\boldsymbol{\alpha}'_{\perp} \boldsymbol{\beta}_{\perp} \right)^{-1} \boldsymbol{\alpha}'_{\perp}$*; and therefore there exists* $K(N)$ *such that* $\left\| \mathbf{\Phi}^{\ell} \mathbf{\Phi}'^{\ell} \right\|_r < K(N)$ *for any* $\ell \in \mathbb{N}$*. Condition (A.6) of Assumption 9 establishes that the sequence of upper bounds* $\{K(N)\}_{N=1}^{\infty}$ *is bounded in* N*. This condition is particularly responsible for convergence result (A.17) below.*

Remark 9 $\boldsymbol{\zeta}_t$ *are assumed to be serially uncorrelated only due to simplicity of exposition.* $\boldsymbol{\zeta}_t$ *could be any stationary process with absolute summable autocovariances uniformly bounded in* N*.*

Recursive substitution of model (A.1) and multiplying by $\mathbf{W}'_i$ from the left yields

$$\mathbf{y}^*_{it} = \mathbf{W}'_i \sum_{\ell=0}^{t-1} \mathbf{\Phi}^{\ell} \boldsymbol{\zeta}_{t-\ell} + \mathbf{W}'_i \mathbf{\Phi}^t \mathbf{y}_0. \tag{A.7}$$

Note that $E\left(\mathbf{y}^*_{it}\right) = \mathbf{0}$ and its variance is bounded by

$$\left\| Var\left(\mathbf{y}^*_{it}\right) \right\|_r \leq \left\| \mathbf{W}_i \right\|_r \left\| \mathbf{W}_i \right\|_c \varrho\left(\mathbf{\Omega}_{\zeta}\right) \sum_{\ell=0}^{t-1} \left\| \mathbf{\Phi}^{\ell} \mathbf{\Phi}'^{\ell} \right\|_r + \left\| \mathbf{W}_i \right\|_r \left\| \mathbf{W}_i \right\|_c \left\| \mathbf{\Phi}^t \mathbf{\Phi}'^t \right\|_r \left\| \mathbf{\Sigma}_0 \right\|_r \tag{A.8}$$

where we have used Rayleigh-Ritz theorem (see Horn and Johnson, 1985, p. 176) and the fact that the spectral radius is a lower bound for any matrix norm (see Horn and Johnson, 1985, Theorem 5.6.9). But under Assumptions 7-9, $\varrho\left(\mathbf{\Omega}_{\zeta}\right) = O(1)$, $\left\| \mathbf{\Phi}^{\ell} \mathbf{\Phi}'^{\ell} \right\|_r = O(1)$ for any $\ell \in \mathbb{N}$, $\left\| \mathbf{\Sigma}_0 \right\|_r = O(1)$. Since $\mathbf{W}_i$ satisfies the granularity condition (6)-(7), it follows that $\left\| \mathbf{W}_i \right\|_r \left\| \mathbf{W}_i \right\|_c = O\left(N^{-1}\right)$,[24] and

$$\left\| Var\left(\mathbf{y}^*_{it}\right) \right\|_r \leq tO\left(N^{-1}\right). \tag{A.14}$$

[23] Two theorems, one due to Engle and Granger (1987) and one due to Johansen (1991, 1996) are both referred to as Granger representation theorem. Johansen provides the moving average representation of a VAR by making assumptions about the autoregressive parameters that characterize the $I(1)$ process. Engle and Granger (1987) state the existence of VECM representation of a process $\mathbf{y}_t$ under the assumptions that $\Delta \mathbf{y}_t$ and $\boldsymbol{\beta}' \mathbf{y}_t$ have stationary and invertible VARMA representation for some full column rank matrix $\boldsymbol{\beta}$. See also Hansen (2005) for a closed form expression for $I(1)$ processes.

[24] Granularity conditions (6)-(7) are equivalent to the following conditions (A.9)-(A.10)

$$\left\| \mathbf{W}_i \right\|_c \neq o(1), \tag{A.9}$$

$$\left\| \mathbf{W}_i \right\|_r = O\left(N^{-1}\right). \tag{A.10}$$

To see this, note that condition (7) implies

$$\frac{\left\| \mathbf{W}_{ij} \right\|^2}{\left\| \mathbf{W}_i \right\|^2} = O\left(N^{-1}\right). \tag{A.11}$$

Hence for any increasing integer valued function $T(N)$ such that

$$\lim_{N\to\infty} \frac{T(N)}{N} = 0, \tag{A.15}$$

we have

$$\lim_{N\to\infty} \sup_{1\leq t\leq T(N)} \|Var(\mathbf{y}_{it}^{*})\| = 0. \tag{A.16}$$

This implies $\mathbf{y}_{it}^{*} \overset{q.m.}{\to} \mathbf{0}$ uniformly in $t \in \{1,..,T\}$ under the asymptotics $N, T \overset{j}{\to} \infty$ such that $T/N \to 0$. Thus we see that the convergence results in Section 3 continues to hold for more general $I(1)$ models with polynomial $\mathbf{\Phi}(L)$ having some of the roots on the unit circle. In particular, note the following approximate error correction representation of $\mathbf{y}_{it}$, under Assumptions 7-9,

$$\lim_{N\to\infty} \sup_{1\leq t\leq T(N)} \|Var(\Delta\mathbf{y}_{it} - (\mathbf{\Phi}_{ii} - \mathbf{I}_{k_i})\mathbf{y}_{i,t-1} - \boldsymbol{\zeta}_{it})\| = 0, \tag{A.17}$$

where $T(N)$ is any increasing integer valued function satisfying equation (A.15). This is because

$$\mathbf{y}_{it} = \mathbf{\Phi}_{ii}\mathbf{y}_{i,t-1} + \mathbf{\Phi}'_{-i}\mathbf{y}_{t-1} + \boldsymbol{\zeta}_{it},$$

and

$$\sup_{1\leq t\leq T} \left\|Var\left(\mathbf{\Phi}'_{-i}\mathbf{y}_{t-1}\right)\right\| = O\left(\frac{T}{N}\right) + O\left(N^{-1}\right)$$

follows from equation (A.14) since $\|\mathbf{\Phi}_{-i}\|_r = O\left(N^{-1}\right)$ by Assumption 9.

B Two Alternative Ways of Introducing Unobserved Common Factors

It is easy to see that VAR model (2) can be considered as a special case of model (4) by letting

$$\mathbf{\Gamma}_g = (\mathbf{\Gamma}_f, -\mathbf{\Phi}_1\mathbf{\Gamma}_f, ..., -\mathbf{\Phi}_p\mathbf{\Gamma}_f), \tag{B.1}$$

and

$$\mathbf{g}_t = \left(\mathbf{f}'_t, \mathbf{f}'_{t-1}, ..., \mathbf{f}'_{t-p}\right)'. \tag{B.2}$$

But $\|\mathbf{W}_i\|^2 = O\left(N^{-1}\right)$ by condition (6). Therefore

$$\|\mathbf{W}_{ij}\|^2 = O\left(N^{-2}\right). \tag{A.12}$$

Since dimension of $\mathbf{W}_{ij}$ is $k_i \times k_j$ (recall that $k_i = O(1)$ for any i), it follows that the order of magnitude of $\|\mathbf{W}_{ij}\|_{\mathfrak{m}}$ does not depend on a particular matrix norm $\|\cdot\|_{\mathfrak{m}}$ under consideration. Hence $\|\mathbf{W}_{ij}\|_r^2 = O\left(N^{-2}\right)$, which prooves equation (A.10). Condition (A.9) can be established by contradiction. Note that equation (A.10) implies that

$$\|\mathbf{W}\|_c = O(1). \tag{A.13}$$

Equations (B.1) and (B.2) imply

$$\mathbf{\Gamma}_g \mathbf{g}_t = \mathbf{\Phi}(L) \mathbf{\Gamma}_f \mathbf{f}_t, \tag{B.3}$$

and model (2) with m_f common factors can always be written as model (4) with up to $m_g = (p+1) m_f$ common factors. In cases when $p = \infty$, model (2) can still be regarded as an approximative special case of model (4) by adopting a finite approximation to the infinite polynomial $\mathbf{\Phi}(L)$.

What about the reverse? In particular, can model (4) be regarded, approximately, as a special case of (2)? Since roots of $|\mathbf{\Phi}(L)| = 0$ lie outside the unit circle, polynomial $\mathbf{\Phi}(L)$ is invertible and we can define new $k \times 1$ dimensional process $\mathbf{h}_t$ as

$$\mathbf{h}_t = \mathbf{\Phi}^{-1}(L) \mathbf{\Gamma}_g \mathbf{g}_t. \tag{B.4}$$

Using the following approximation

$$\mathbf{R}(L, q) = \sum_{\ell=0}^{q} \mathbf{R}_{q,\ell} L^{\ell} \approx \mathbf{\Phi}^{-1}(L) \tag{B.5}$$

yields an approximation of $\mathbf{h}_t$

$$\mathbf{h}_t \approx \mathbf{R}(L, q) \mathbf{\Gamma}_g \mathbf{g}_t = \mathbf{\Gamma}_f \mathbf{f}_t, \tag{B.6}$$

where $\mathbf{\Gamma}_f = (\mathbf{R}_{q0}\mathbf{\Gamma}_g, ..., \mathbf{R}_{qq}\mathbf{\Gamma}_g)$ and $\mathbf{f}_t = \left(\mathbf{g}_t', ..., \mathbf{g}_{t-q}'\right)'$ is a $m_g(q+1) \times 1$ dimensional vector of factors. It follows from equation (B.6) that model (4) can be written as

$$\mathbf{\Phi}(L)(\mathbf{x}_t - \boldsymbol{\delta}_0 - \boldsymbol{\delta}_1 t - \mathbf{\Gamma}_f \mathbf{f}_t) \approx \mathbf{D}\boldsymbol{\xi}_t. \tag{B.7}$$

Thus model (4) is approximately a special case of model (2).[25]

C Lemmas and Proofs

Proof of lemma 1. Using the self-adjoint and submultiplicative properties of the spectral norm, we have

$$\begin{aligned}
\left\| Var\left(\sum_{\ell=2-p}^{t} \Delta\mathbf{y}_{i\ell}^{*}\right) \right\| &= \left\| \sum_{\ell=2-p}^{t} \sum_{k=2-p}^{t} E\left(\Delta\mathbf{y}_{i\ell}^{*} \Delta\mathbf{y}_{ik}^{*\prime}\right) \right\|, \\
&\leq (t-1+p) \left\| Var\left(\Delta\mathbf{y}_{it}^{*}\right) \right\| + 2 \sum_{\ell=1}^{t-2+p} \left\| \mathfrak{C}_{\ell} \right\| (t-1+p-\ell), \\
&\leq (T-1+p) \left(\left\| Var\left(\Delta\mathbf{y}_{it}^{*}\right) \right\| + 2 \sum_{\ell=1}^{\infty} \left\| \mathfrak{C}_{\ell} \right\| \right), \\
&= O\left(\frac{T}{N}\right),
\end{aligned} \tag{C.1}$$

[25] This analysis can be generalized for the case where some of the roots of $|\mathbf{\Phi}(L)| = 0$ lie on the unit circle using Granger's representation theorem.

where $\|Var(\Delta\mathbf{y}_{it}^{*})\| = O\left(N^{-1}\right)$ and $\sum_{\ell=1}^{\infty}\|\mathfrak{C}_{\ell}\| = O\left(N^{-1}\right)$ as established in Lemma 2. ■

Lemma 2 *Consider model (2) and let Assumptions 1-3 hold. Then for any weights* $\mathbf{W}_i$ *satisfying condition (6) only,*

$$E\left(\Delta\mathbf{y}_{it}^{*}\right) = \mathbf{0}, \tag{C.2}$$

$$\sup_{1\le t\le T}\|Var\left(\Delta\mathbf{y}_{it}^{*}\right)\| = O\left(N^{-1}\right), \tag{C.3}$$

as well as

$$\sum_{\ell=1}^{\infty}\|\mathfrak{C}_{\ell}\| = O\left(N^{-1}\right), \tag{C.4}$$

where $\mathfrak{C}_{\ell} = E\left(\Delta\mathbf{y}_{i,t+\ell}^{*}\Delta\mathbf{y}_{it}^{*\prime}\right)$ *is* ℓ*-th autocovariance of* $\Delta\mathbf{y}_{it}^{*}$*, and* $\mathbf{y}_{it}^{*} \equiv \mathbf{W}_i'\mathbf{y}_t$*.*

Proof of lemma 2. Recall that $\Delta\mathbf{y}_t$, under Assumption 1-3, is

$$\Delta\mathbf{y}_t = \mathbf{R}(L)\Delta\mathbf{u}_t = \mathbf{R}(L)\mathbf{D}\mathbf{\Psi}(L)\mathbf{v}_t = \mathbf{C}(L)\mathbf{v}_t, \tag{C.5}$$

where

$$\mathbf{C}(L) = \sum_{\ell=0}^{\infty}\mathbf{C}_{\ell}L^{\ell} = \mathbf{R}(L)\mathbf{D}\mathbf{\Psi}(L),\ \mathbf{\Psi}(L) = \begin{pmatrix} \mathbf{\Psi}_0(L) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Psi}_N(L) \end{pmatrix},\ \mathbf{v}_t = \begin{pmatrix} \mathbf{v}_{0t} \\ \vdots \\ \mathbf{v}_{Nt} \end{pmatrix},$$

and $\mathbf{C}_k = \sum_{\ell=0}^{k}\mathbf{R}_{\ell}\mathbf{D}\mathbf{\Psi}_{k-\ell}$.

$$E\left(\mathbf{W}_i'\Delta\mathbf{y}_t\right) = \mathbf{W}_i'\mathbf{C}(L)E(\mathbf{v}_t) = 0, \tag{C.6}$$

$$\begin{aligned} \|Var\left(\Delta\mathbf{y}_{t}^{*}\right)\| &= \left\|\mathbf{W}_i'Var\left(\Delta\mathbf{y}_t\right)\mathbf{W}_i\right\|, \\ &\le \left\|\mathbf{W}_i'\right\|\|Var\left(\Delta\mathbf{y}_t\right)\|\|\mathbf{W}_i\|, \\ &\le \|\mathbf{W}_i\|^2\sum_{\ell=0}^{\infty}\left\|\mathbf{C}_{\ell}Var(\mathbf{v}_t)\mathbf{C}_{\ell}'\right\|, \\ &\le \|\mathbf{W}_i\|^2\|\mathbf{\Omega}_v\|\sum_{\ell=0}^{\infty}\|\mathbf{C}_{\ell}\|^2, \end{aligned} \tag{C.7}$$

where we have used submultiplicative property of matrix norms, the fact that the spectral norm is self-adjoint (see Horn and Johnson, 1985, p. 309), and

$$\mathbf{\Omega}_v = \begin{pmatrix} \mathbf{\Omega}_{v0} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Omega}_{vN} \end{pmatrix}.$$

Note that $\|\mathbf{\Omega}_v\| \le \sqrt{\|\mathbf{\Omega}_v\|_r\|\mathbf{\Omega}_v\|_c}$ (See Bernstein, 2005, p. 369, Fact 9.8.15). Since $\mathbf{\Omega}_v$ is symmetric, it follows

$$\|\mathbf{\Omega}_v\| \le \|\mathbf{\Omega}_v\|_r = \max_{i\in\{0,1,..,N\}}\|\mathbf{\Omega}_{vi}\|_r = O(1) \tag{C.8}$$

by Assumption 2.

$$\begin{aligned}
\sum_{k=0}^{\infty}\|\mathbf{C}_k\| &= \sum_{k=0}^{\infty}\left\|\sum_{\ell=0}^{k}\mathbf{R}_\ell\mathbf{D}\boldsymbol{\Psi}_{k-\ell}\right\|, \\
&\leq \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\|\mathbf{R}_\ell\|\,\|\mathbf{D}\|\,\|\boldsymbol{\Psi}_{k-\ell}\|, \\
&\leq \|\mathbf{D}\|\sum_{\ell=0}^{\infty}\|\mathbf{R}_\ell\|\cdot\sum_{\ell=0}^{\infty}\|\boldsymbol{\Psi}_\ell\|.
\end{aligned}$$

$\|\boldsymbol{\Psi}_\ell\|$ is bounded by $\|\boldsymbol{\Psi}_\ell\| \leq \sqrt{\|\boldsymbol{\Psi}_\ell\|_r\|\boldsymbol{\Psi}_\ell\|_c} \leq \sqrt{\max_i\|\boldsymbol{\Psi}_{\ell i}\|_r \max_i\|\boldsymbol{\Psi}_{\ell i}\|_c}$. Since $\|\boldsymbol{\Psi}_{\ell i}\|_c \leq k_i\|\boldsymbol{\Psi}_{\ell i}\|_r$ (See Horn and Johnson, 1985, p. 314), we have

$$\sum_{\ell=0}^{\infty}\|\boldsymbol{\Psi}_\ell\| \leq \left(\max_{i\in\{0,1,..,N\}}\sqrt{k_i}\right)\cdot\sum_{\ell=0}^{\infty}\max_{i\in\{0,1,..,N\}}\|\boldsymbol{\Psi}_{\ell i}\|_r = O(1) \tag{C.9}$$

by Assumptions 1-2. Furthermore, $\|\mathbf{D}\| \leq \sqrt{\|\mathbf{D}\|_r\|\mathbf{D}\|_c} = O(1)$ and $\sum_{\ell=0}^{\infty}\|\mathbf{R}_\ell\| = O(1)$ by Assumption 1. It follows

$$\sum_{k=0}^{\infty}\|\mathbf{C}_k\| = O(1). \tag{C.10}$$

This implies

$$\sum_{k=0}^{\infty}\|\mathbf{C}_k\|^2 = O(1). \tag{C.11}$$

Noting that $\|\mathbf{W}_i\|^2 = O\left(N^{-1}\right)$ by condition (6), equations (C.7)-(C.8),(C.11) establish

$$\|Var(\Delta\mathbf{y}_t^*)\| = \sup_{1\leq t\leq T}\|Var(\Delta\mathbf{y}_t^*)\| = O\left(N^{-1}\right). \tag{C.12}$$

This completes the proof of result (C.3). In order to establish result (C.4), note that

$$\begin{aligned}
\sum_{\ell=1}^{\infty}\|\mathfrak{C}_\ell\| &\leq \|\mathbf{W}_i\|^2\sum_{\ell=1}^{\infty}\sum_{k=0}^{\infty}\|\mathbf{C}_{k+\ell}\|\,\|\mathbf{C}_k\|\,\|\boldsymbol{\Omega}_v\|, \\
&\leq \|\mathbf{W}_i\|^2\|\boldsymbol{\Omega}_v\|\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\|\mathbf{C}_k\|\,\|\mathbf{C}_{k+\ell}\|, \\
&\leq \|\mathbf{W}_i\|^2\|\boldsymbol{\Omega}_v\|\sum_{k=0}^{\infty}\|\mathbf{C}_k\|\cdot\sum_{k=0}^{\infty}\|\mathbf{C}_k\|, \\
&= O\left(N^{-1}\right),
\end{aligned} \tag{C.13}$$

where we have used equations (C.8),(C.10) and as before, $\|\mathbf{W}_i\|^2 = O\left(N^{-1}\right)$ by condition (6). This completes the proof. ■

Lemma 3 *Let* $\mathbf{V} = (\mathbf{V}_0, \mathbf{V}_1, ..., \mathbf{V}_N)'$ *be any pre-determined* $k\times k_v$ *dimensional matrix satisfying* *following conditions.*

$$\begin{aligned}\left\|\mathbf{V}_w\right\|_r &= O\left(N^{-1}\right), & \text{(C.14)}\\ \left\|\mathbf{V}\right\|_r &= O\left(1\right), & \text{(C.15)}\end{aligned}$$

where $\mathbf{V}_w = (\mathbf{0}_{k_v \times k_0}, \mathbf{V}_1, ..., \mathbf{V}_N)'$. *Then*

$$\sum_{\ell=0}^{\infty} \mathbf{V}'\mathbf{\Phi}^{\ell}\mathbf{D}_w \Delta\boldsymbol{\xi}_{t-\ell} \overset{q.m.}{\to} 0, \tag{C.16}$$

as $N \to \infty$, *where matrix* $\mathbf{\Phi}$ *and* $\mathbf{D}_w$ *satisfies Assumption 5, and* $\Delta\boldsymbol{\xi}_t$ *is given by Assumption 2.*

Proof. Lemma 3 is a direct extension of Lemma 2 presented in supplement of Chudik and Pesaran (2011) to multivariate case. ■