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An Alternative Neo-Classical Growth Model with Closed–Form Decision Rules

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ABSTRACT

A version of a representative agent model is constructed in which closed-form decision rules are produced for rather general production technologies. Agents trade in capital, and the decision rules can be used to characterize the volume of this trade.

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It is generally asserted that versions of the neo-classical growth model that admit closed form decision rules for consumption and investment are rather special. In particular, it can be shown that these decision rules can be constructed in the case which either i) the utility function is logarithmic and the production function is Cobb-Douglas with 100% depreciation, or ii) the utility function is of the constant relative risk-aversion variety, and the production function is linear in capital (see Danthine and Donaldson (1981) for a proof). In particular, it is not possible to arrive at closed-form decision rules for the case in which there is a C.E.S. production technology, or with a depreciation rate greater than zero. This may be thought to be unfortunate since this is a relatively small subset of economies that are of interest. The present note shows that making a small modification to the usual neo-classical growth model enables the model to yield closed-form decision rules. With this modification, the resulting model has the traditional infinitely-lived representative agent model, and the two period-lived overlapping generations models as special cases. It is also shown that it is possible to characterize the stochastic volume of trade in capital, which is non-trivial in the model.

ECONOMIC ENVIRONMENT

The economy is one in which time is discrete, and indexed by t=1,2,3... Initially at date t=1 there are a continuum of agents in the economy, and this population is normalized to size unity. For convenience, the population size is normalized to unity. For all agents that are in the economy at date t, their preferences can be described by the following utility function

$$E\left[\sum_{s=t}^{\infty}\beta^{s-t}\log(c_s)\right], \tag{1}$$

where $\beta \in (0, 1)$. At each date t, αN of the existing agents leave the economy, and αN "new" agents enter the economy where $\alpha \in (0, 1)$. Agents know at the beginning of a period whether or not they must leave the economy at the end of that period. In any period t, the probability that an agent will have to leave the economy in the following period is α, and this probability is identical for all agents and for all periods. In other words, the probability that an agent who has been present for only one period will leave, is equal to that probability for an agent who had been present for many periods.

Agents who are new entrants to the economy are endowed with a unit of labor effort that can be supplied to produce output. Agents have this endowment of labor effort only in their first period and supply it inelastically. These agents then receive labor income in the first period of their life, and can then try to save some of this for consumption in future periods.

Agents who are in the economy at the initial date t=1, hold, in the aggregate, K_1 units of capital. This capital depreciates at the rate δ per period.[1] In general, if K_t units of capital are employed in production in period t, then it can be supplied to produce output (Y_t) according to the technology

$$Y_t = \lambda_t[\theta l_t^{(-1/\rho)} + (1-\theta)K_t^{(-1/\rho)}]^{-\rho}, \tag{2}$$

where l_t is the amount of labor employed, and λ_t is a random technological disturbance. Since there are α agents entering the economy, with each supplying a single unit of labor effort, in equilibrium it must be that $l_t = \alpha$.[2] Of course, the CES production function is meant to have the Cobb-Douglas technologies as special cases.[3] Now the state variables for the economy are K_t, and λ_t, which are known at the beginning at date t. The consumption good can be converted into capital at a one-for-one basis.

The economy evolves as follows. Agents who enter the economy in period t, work to produce output, and get paid their marginal product. With the resulting income, these agents consume, and also produce capital that will finance future consumption. In future periods, agents supply capital for production, and consequently receive the marginal product of capital as income. They then decide how much to consume and to save (invest). Henceforth, we let r_t and w_t, denote the marginal product of capital and labor in period t.

The dynamic programming problem faced by an agent who currently has y_t units of wealth at the beginning of period t is the following[4]

$$V(y_t) = \max\{\log(y_t - k_{t+1}) + \beta E[(1-\alpha)V(r_{t+1}k_{t+1}) + \alpha\log(r_{t+1}k_{t+1})]\} \tag{3}$$

Obviously, this reflects the fact that with probability α the agent will leave the economy in the following period, and with probability $(1-\alpha)$, the agent will not leave. After some manipulation, it is possible to verify that the optimal decision rule associated with this problem is the following

$$k_{t+1} = \left[\frac{\beta}{1+\alpha\beta}\right]y_t. \tag{4}$$

As usual with logarithmic preferences, the agent saves a constant fraction of income. Hence, the level of savings or investment is a decreasing function of α, the rate at which agents leave the economy, but an increasing function of β, the discount factor. The higher is α, the shorter is the expected horizon over which the agent's optimization problem takes place, and hence the less they will wish to save for future consumption. When $\alpha = 0$, the savings rate is equal to the discount factor (β), which conforms with what is known about economies in which the technology is constant returns, and preferences are logarithmic. Alternatively, when $\alpha = 1$, the model is that of two period-lived overlapping generations. When $\alpha \in (0, 1]$, the expected lifetime, or number of periods over which the agent expects to exist in the economy is $(\alpha+1)/\alpha$.

For capital holders who were present in the economy in period t-1, and entered period t with k_t units of capital, the agents wealth is then $y_t = r_t k_t$. Here r_t equals the marginal product of capital, which is determined as follows

$$r_t = \lambda_t(1-\theta)K_t^{\frac{-1-\rho}{\rho}}\left[\theta\alpha^{-1/\rho} + (1-\theta)K_t^{-1/\rho}\right]^{-\rho-1} + (1-\delta). \tag{5}$$

For laborers who are new entrants to the economy, their wealth is then determined as follows

$$y_t = \lambda_t\theta\left[\alpha^{\frac{-1-\rho}{\rho}}\right]\left[\theta\alpha^{-1/\rho} + (1-\theta)K_t^{-1/\rho}\right]^{-\rho-1}. \tag{6}$$

For new entrants to the economy, they supply their unit of labor effort, and then their initial level of income in the first period of their life is this labor income. They then save a constant fraction of this income. For existing entrants, they supply their capital and their income in any period is the capital income received at the beginning of that period. Therefore, since all agents save a fraction of their income, and with constant returns to scale production technology, it follows that the capital stock then obeys the following law of motion[5]

$$K_{t+1} = \left[\frac{\beta}{1+\alpha\beta}\right]\left[\lambda_t[\theta\,\alpha^{(-1/\rho)} + (1-\theta)K_t^{(-1/\rho)}]^{-\rho} + (1-\delta)K_t\right]. \tag{7}$$

It is now of interest to inquire about the volume of trade in capital in this framework. Fortunately, this is easy to characterize. It is possible to define the volume of trade as the change in the holdings of all sellers (or, alternatively, buyers) of capital. In this case the volume of trade is said to be

$$\left[\frac{(1-\alpha)}{2}\right]\left|\left[\frac{\beta}{1+\alpha\beta}\right]\left[\lambda_t[\theta\,\alpha^{(-1/\rho)} + (1-\theta)K_t^{(-1/\rho)}]^{-\rho} + (1-\delta)K_t\right] - K_t\right| + (\alpha)K_t/2. \tag{8}$$

This reflects the fact that α of the agents must sell <u>all</u> their capital, and this is reflected in the second part of the above equation. The first part of the equation reflects the behavior existing agents and new entrants. It is easy to analyze various fiscal policies in this environment. For example, on can analyze the impact of a capital income tax on this environment. The impact that this has on the volume of trade in capital is shown in Table 1. In this case the following parameter values are chosen as follows: α=.08, β=.96, θ=.7, ρ=2, δ=.10, and log(λ_t) is i.i.d. and distributed as N(0,.08). A higher tax rate raises the volatility of and lowers the average transaction volume.

FURTHER REMARKS

By utilizing this version of the representative agent framework, it may be possible for other researchers to address other issues in a more illuminating manner. Huffman (1992) has already employed this model to analyze how a capital gains tax, or a tax on financial transactions could influence the behavior of financial market variables, including the volume of trade. It is also possible to show that this model generates an endogenous <u>distribution</u> of capital holdings, and that various fiscal policies can then affect this distribution. Further research is currently being conducted as to how similar policies would influence the level of growth and capital accumulation in a related framework. Presumably it would be possible to utilize similar environments to analyze the impact of various tax or fiscal policies.

One might wish to incorporate an endogenous labor/leisure choice. To this end, it might be assumed that the utility function is of the following form

$$E\left[\sum_{s=t}^{\infty}\beta^{s-t}\log(c_s)\right] + \log(1 - n_t) \tag{9}$$

where the expectation operator reflects the probability α of having to leave the economy in any period and n_t is the amount of labor effort in period t. After setting up the dynamic programming problem, it is easily seen that the optimal labor effort is determined as $n_t = (1+\alpha\beta)/(2+\alpha\beta)$. One might naturally inquire as to how the decision rules would be altered if the agent had preferences determined according to the constant relative risk aversion variety:

$$E\left[\sum_{s=t}^{\infty}\beta^{s-t}(c_s)^{1-\rho}\right], \tag{10}$$

for $\rho > 0$, $\rho \neq 1$. In this case, it is easy to show that the savings rate is determined according to the equation

$$s^{\rho} = \beta[(1 - \alpha) + \alpha(1 - s)^{\rho}]E_t(r_{t+1}^{1-\rho}). \tag{11}$$

Of course, in this instance the savings rate is increasing in term $E_t(r_{t+1}^{1-\rho})$.

Table 1

Capital Tax Rate	0%	5%	10%
Average transaction volume per period	.4253	.3672	.3247
% standard deviation of transaction volume	16.95%	19.44%	21.76%

FOOTNOTES

1. In what follows, upper case letters denote economy-wide totals, and lower case letters represent individual variables.

2. It does not have to be that the same number of agents leave and enter the economy at the same time. However, this assumption makes these disturbances idiosyncratic, and thereby does not permit this exogenous to produce aggregate shocks.Obviously it would be possible to let α be a stochastic variable, and this would then permit the exogenous labor supply fluctuate as well.

3. The technology shock could enter the production function in a different manner, but this is not important for what is to follow.

4. It is easy to show that the value function turns out to be of the following form: $V(y_t) = \pi_0 + \pi_1\log(y_t)$, for some constants π_0 and π_1.

5. Up till now it has been implicit that the capital stock could be converted into the consumption good, and hence that one could ignore the "corner" associated with the optimization problem. If this is not ignored, it easy to show that the evolution of the capital stock in this instance is determined instead according to the equation $K_{t+1}^* = \max\{(1 - \delta)K_t^*, K_{t+1}\}$, where K_{t+1} is given by the right side of equation (4). To simplify matters, the remainder of the paper will focus on the case in which this corner is never binding.

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