A General Two Sector Model of Endogenous Growth
with Human and Physical Capital

by

Eric W. Bond, Ping Wang, and Chong K. Yip

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Eric W. Bond*, Ping Wang**, and Chong K. Yip***

*Eric W. Bond, Department of Economics
The Pennsylvania State University

**Ping Wang, Research Department
Federal Reserve Bank of Dallas

***Chong K. Yip, Department of Economics
Georgia State University

I. Introduction

The purpose of this paper is to present a two sector, two factor model
of endogenous growth in which physical capital is accumulated in a goods
sector and human capital is accumulated in an education sector. This model
thus follows in the spirit of Lucas (1988), who assigned a central role to the
process of human capital accumulation in the determination of the rate of
economic growth. We generalize the Lucas model, which assumed that labor time
alone was required to accumulate human capital, by allowing for the education
sector to use inputs of both physical and human capital. We derive a
condition under which this model has a balanced growth path in which both
types of capital and consumption grow at a common rate, and show that the
system will have a saddle path along which it converges to the balanced growth
path. We then illustrate the role played by the factor intensities of the two
sectors in determining the transitional dynamics of the model and the
comparative static effects of parameter changes.

A central feature of our analysis is the use of results from two sector
models of international trade theory (e.g., Jones (1965)) to simplify the
dynamics of the model and obtain results on transitional dynamics for general
functional forms. Previous work in models of this type has relied on
simulation analysis to analyze the effects of policy changes (e.g., King and
Rebelo (1990)), or has restricted attention to the solvability of the balanced
growth rate (e.g., Rebelo (1991)). In particular, we emphasize the role
played by the intertemporal no-arbitrage condition which requires an equality
between the rates of return on physical and human capital. The dynamic
optimization problem yields the result that the relative price of education
output will equal the value of an increment of human capital (the ratio of the

respective costate variables) in the optimal program. The intertemporal no-
arbitrage condition specifies the change in the value of a unit of human
capital which is necessary to equalize the returns (net of depreciation) of
the two investments. Due to the two sector nature of the technology, the wage
rate and rental on capital are determined by the relative price of a unit of
human capital (independently of the factor supplies). An increase in the
price of education will raise the return to the factor used intensively in
that sector and reduce the return to the other factor (the Stolper-Samuelson
theorem). The no-arbitrage condition thus becomes a differential equation
determining the evolution of the relative price of human capital, which can be
used to determine the value of a unit of human capital on the balanced growth
path (where the relative price of human capital is constant), and to determine
the dynamic adjustment of the value of human capital in the neighborhood of
the balanced growth value.[1]

We show that if the education sector is labor intensive relative to
goods production, the adjustment process for the value of a unit of human
capital is stable. However, if the education sector is capital-intensive,
the adjustment process is unstable. The saddle path for the latter case
requires that the price of human capital jump to its balanced growth path
value and remain at that level throughout the process. A similar potential
for instability exists with respect to the quantity adjustment process in the
former case. Due to the two sector nature of the technology, an increase in
the capital labor ratio results in an increase in the output of the capital-
intensive good and a decrease in the output of the labor intensive good at
constant prices (the Rybczynski theorem). In the case where the goods sector
is capital intensive, capital accumulation results in a decrease in the output

of the education sector, further increasing the capital-labor ratio. This adjustment process would be unstable at fixed prices, so it is necessary for the price of education output to rise along the transition path in order for this process to converge to the capital-labor ratio consistent with balanced growth. Thus for each of the factor intensity rankings, an unstable adjustment process (prices or quantities) is offset by adjustments in the other variable to obtain a saddle path.[2]

The no-arbitrage condition is also useful for analyzing the effects of parameter changes and policy changes on the balanced growth path. For example, we compare the effects of a tax on capital income with a tax on labor income, assuming that labor in the educational sector is untaxed. The tax on capital income reduces the net return to capital, so the price of educational output must adjust to raise the gross return of capital relative to labor. This requires an increase in the price of education iff the education sector is capital intensive. A tax on labor income, on the other hand, must result in a decline in the relative price of education output. The factor intensity ranking does not determine the direction of the price change in the case of a labor income tax, because the labor tax falls only on labor in the goods sector. It thus has an effect similar to that of a tax on goods sector output. We also show that both of these factor taxes must result in a decline in the growth rate, because both lead to a decline in the net of tax return to capital accumulation. An increase in the capital tax depresses the growth rate by more than an equal percentage increase in the labor income tax, because of the broader coverage of the capital tax.

Section II of the paper presents the two sector model. Section III characterizes the balanced growth path, and establishes the conditions under

which a balanced growth path will exist. Section IV establishes that the
system will have a saddle path, and analyzes the transitional dynamics in the
neighborhood of the balanced growth path. Section V concludes with some
applications of the model illustrating the effects of parameter changes and
factor taxation on the growth rate and the balanced growth path.

II. A Two-Sector Model of Endogenous Growth

In this section we present a two factor, two sector model of endogenous
growth in which there is a goods sector, X, and an education sector, Y. In
each sector, output is produced using capital and labor under conditions of
constant returns to scale. The output of the education sector raises the
supply of effective labor units by adding to the stock of human capital, while
the output of the goods sector may either be consumed or added to the stock of
physical capital. Since the model exhibits constant returns to scale in
reproducible inputs, the equilibrium will be characterized by perpetual growth
as analyzed by Rebelo (1991).

Let $K(t)$ denote the aggregate stock of physical capital at time t and
$H(t)$ the aggregate stock of effective labor units (human capital). Factors
are assumed to be fully mobile across sectors at a point in time, and $S_{ix}(t)$
is the share of the stock of factor i (i=h,k) that is allocated to sector X at
time t. Under the assumption of constant returns to scale, the production
technology can be represented as

$$X = F(S_{kx} K, \ S_{hx} H) = S_{hx} Hf(k_x)$$
$$Y = G((1-S_{kx})K, \ (1-S_{hx})H) = (1-S_{hx})Hg(k_y),$$

where $k_x = [(S_{kx} K)/(S_{hx} H)]$ and $k_y = [(1-S_{kx})K]/[(1-S_{hx})H]$ are the

capital/labor ratios in the respective sectors. The output per unit of (effective) labor functions f and g are increasing and strictly concave.

Output in the goods sector can be either used as consumption [C(t)] or added to the physical capital stock. Letting δ be the rate of depreciation of physical capital, the evolution of that capital stock can be expressed as

$$\dot{K} = S_{hx}Hf(k_x) - \delta K - C. \tag{1}$$

We simplify the treatment of labor by assuming that there is a fixed population with an (inelastically supplied) stock of time endowment. The only changes in the effective labor force will result from changes in human capital per worker, which is assumed to be a perfect substitute for a change in the size of the labor force. Letting η be the depreciation rate of human capital, the change in the stock of human capital will be[3]

$$\dot{H} = (1-S_{hx})Hg(k_y) - \eta H. \tag{2}$$

In what follows, we will use the terms labor and human capital interchangeably. Note that the educational process is assumed to require both physical capital (libraries, laboratories, etc.) and human capital (faculty and students). By measuring the labor input in effective units, we are assuming that workers with more human capital teach more effectively and learn more quickly. This differs from the specification in Lucas (1988), where human capital accumulation requires only an input of time.[4]

The representative agent is assumed to have time-separable preferences with a constant subjective rate of time preference (ρ) and an instantaneous utility function $U = C^{1-\sigma}/(1-\sigma)$. This utility function exhibits constant intertemporal elasticity of substitution (the inverse of this elasticity is

denoted by σ, where $0 < \sigma < \infty$), and is chosen because it yields a balanced growth path.[5] Given all preference and technology parameters, the representative agent's optimization problem can be specified as

$$V(H_0, K_0) = \max_{C, S_{hx}, S_{kx}, K, H} \int_0^\infty \frac{C(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \qquad (P1)$$

subject to (1), (2), $H(0) = H_0$, $K(0) = K_0$, and non-negativity constraints on all quantities. (P1) is a dynamic optimization problem with control variables C, S_{hx}, and S_{kh} and state variables H and K. The utility function is concave in the controls, and the technology is convex. If $\rho > (1-\sigma)\nu_{max}$, where ν_{max} is the highest attainable rate of consumption growth, then $V(K,H) < \infty$ and the results of Benveniste and Scheinkman (1982) can be applied to yield necessary conditions for the optimal growth path. Forming the current value Hamiltonian for this problem, with μ and λ as the costate variables associated with K and H respectively, we obtain the following necessary conditions:

$$C^{-\sigma} - \mu = 0 \qquad (3a)$$

$$(\mu f' - \lambda g')K = 0 \qquad (3b)$$

$$[\mu(f - k_x f') - \lambda(g - k_y g')]H = 0 \qquad (3c)$$

$$\dot{\mu} = \mu(\rho + \delta) - \mu S_{kx} f' - \lambda(1 - S_{kx})g' \qquad (3d)$$

$$\dot{\lambda} = \lambda(\rho + \eta) - \mu S_{hx}(f - k_x f') - \lambda(1 - S_{hx})(g - k_y g') \qquad (3e)$$

$$\lim_{t \to \infty} e^{-\rho t} \mu(t)K(t) = 0 \qquad (3f)$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t)H(t) = 0, \qquad (3g)$$

together with equations (1) and (2).[6]

In the next section, we will analyze the necessary conditions given by

(3), and show that there exists a balanced growth path along which C, H, and K grow at a common rate. Furthermore, along this balanced growth path μ and λ grow at a common rate. Since the costate variables represent the values of a unit of human and physical capital respectively, the ratio of these variables $p = \lambda/\mu$ is the relative value of a unit of human capital, which will also be constant on the balanced growth path. We then show in Section IV that the model has a saddle path which converges to this balanced growth path.

III. The Balanced Growth Path

In analyzing the necessary conditions (3), we will make use of the fact that the use of the relative price p in this two sector growth model yields substantial simplifications of the type that are familiar from the analysis of the Heckscher-Ohlin model of international trade. The two factor, two good nature of the technology means that this model has the factor price equalization property: prices of the productive factors (and therefore the factor proportions k_x and k_y) can be solved as functions of the relative output prices alone when both sectors are in operation. This property makes it possible to analyze the dynamics of the relative price of human capital separately from the other variables in the system. Therefore, we will first analyze the behavior of p, and then turn to the evolution of the remaining variables of the system.

A. Output Prices, Factor Returns, and Steady-state Growth

In this section we show that the factor prices and factor proportions are determined by the relative price of Y output along the optimal path as long as both sectors are in operation.[7] We then use the intertemporal no-

arbitrage conditions, which guarantee the equalization of returns from investment in physical and human capital, to derive the evolution of this relative price of Y output along the optimal path. We show that evolution of p and the growth rate of consumption are both determined by the level of p alone.

Equations (3b) and (3c) require that the marginal revenue products of capital and labor be equalized across sectors at each point in time, where marginal revenue products are evaluated using the price of human capital, p. Defining $r_j(k_j)$ to be the marginal physical product of physical capital in sector j ($j=x,y$) and $w_j(k_j)$ the marginal physical product of human capital, conditions (3b) and (3c) can be written as

$$r_x(k_x) = pr_y(k_y) \qquad \qquad (4a)$$

$$w_x(k_x) = pw_y(k_y). \qquad \qquad (4b)$$

Equations (4) can be solved for the equilibrium factor proportions k_j (or equivalently the equilibrium factor returns) as functions of p alone. Totally differentiating (4) yields

$$dk_x/dp = (f''g'')^{-1} (r_y w_y' - w_y r_y')/(k_x - k_y)$$

$$dk_y/dp = (f''g'')^{-1} (r_y w_x' - w_y r_x')/(k_x - k_y).$$

By the definition of the marginal productivities, $r_x' = f'' < 0$, $r_y' = g'' < 0$, $w_x' = -k_x f'' > 0$ and $w_y' = -k_y g'' > 0$. The first two expressions in parentheses must be positive in each of the above equations, which means that the relation between p and factor proportions is determined by the relative factor intensities of the two sectors. This can be summarized as:

Lemma 1: The capital/labor ratios in the optimal solution can be expressed as

$k_x(p)$ and $k_y(p)$. Capital/labor ratios are increasing in p iff the education sector is labor intensive relative to the goods sector $(k_x > k_y)$.

This is an application of the Stolper-Samuelson theorem, which states that an increase in the relative price of a good causes an increase in the relative reward of the factor used intensively in that sector. When the education sector is labor (capital) intensive, an increase in its price raises (lowers) the relative cost of labor, which will in turn raise the capital/labor ratio in both sectors.

In a balanced growth equilibrium, the shadow values λ and μ must grow at the same rate, so that p is a constant. Subtracting (3c) from (3d) and utilizing Lemma 1, we obtain an expression for the evolution of p

$$\frac{\dot{p}}{p} = r_x(k_x(p)) - w_y(k_y(p)) + \eta - \delta. \tag{5}$$

Equation (5) is an intertemporal no-arbitrage condition: it specifies the intertemporal relative-price adjustment necessary to equalize the net returns on human and physical capital. If $r_x - \delta > w_y - \eta$, the rental value of capital (net of depreciation) exceeds that on human capital, and there must be a capital gain earned on human capital investments to offset the difference in net rental values if investment in both types of capital is to take place.

The relative price of sector y output in a balanced growth equilibrium will be the value that solves (5) with $\dot{p}/p = 0$. Note that this price will depend on the depreciation rates of the two types of capital and the marginal productivity of the factors, but not on the rate of time preference. Since the evolution of the price of human capital is a function of p alone, we can also use this to examine the stability of the relative price adjustment

process. Differentiating yields

$$d(\frac{\dot{p}}{p}) = [r_x{'}(\frac{dk_x}{dp}) - w_y{'}(\frac{dk_y}{dp})]dp. \qquad (6)$$

Since $r_x{'} < 0$ and $w_y{'} > 0$, application of Lemma 1 indicates that the expression in brackets on the right hand side of (6) is negative iff $k_x > k_y$. Thus, we have:

Lemma 2: The intertemporal adjustment process of the relative price of human capital investment is a function of p alone. This process is stable ($d\dot{p}/dp < 0$) iff the education sector is labor intensive relative to the goods sector ($k_x > k_y$).

The role of the sectoral factor intensities in the dynamics of p is illustrated in Figure 1. The $\dot{p}/p = 0$ locus must be downward-sloping in (k_x,k_y) space, since the difference between returns to capital and labor is decreasing in the capital intensity of both sectors. Points above the $\dot{p}/p = 0$ locus must then be associated with $\dot{p}/p < 0$. The $\phi = r_x w_y - r_y w_x = 0$ locus in Figure 1 describes competitive equilibrium factor usage, which is the locus of values of sectoral capital intensities that are consistent with equalization of factor returns across sectors [(4a) and (4b)] at various values of p. By Lemma 1, this locus must be upward-sloping. Suppose we start from an initial value of p at which $\dot{p}/p < 0$, which occurs when the net rental on human capital exceeds that on physical capital. If $k_x > k_y$, the reduction of p will raise the relative return to physical capital and reduce the capital/labor ratios in both sectors (by Lemma 1). This reduces the differential between physical and human capital, resulting in movement along the $\phi = 0$ locus toward point E_1 on the $\dot{p}/p = 0$ as illustrated in Figure 1a.

The price adjustment process is stable in this case.

If $k_x < k_y$, the price adjustment process is unstable. Starting from a point where $\dot{p}/p < 0$, the reduction in p will reduce the relative return to capital, which raises capital/labor ratios and widens the gap between the return to physical and human capital as shown in Figure 1b. In this case, the adjustment process can only be completed if the system jumps immediately to the steady-state price level given by point E_2 in Figure 1b.

The current level of p will also determine the growth rate of consumption. Differentiating condition (3a) for optimal consumption with respect to t and equating to (3d) yields

$$\dot{C} = \nu_C(p)C, \tag{7a}$$

$$\text{where } \nu_C = \sigma^{-1} \left[r_x(k_x(p)) - (\rho+\delta) \right]. \tag{7b}$$

This condition requires that the marginal utility of consumption decline at the same rate as the shadow value of a unit of capital, $\dot{\mu}/\mu$, along the optimal path. Returns from consumption will be equalized with those from investing in capital. Differentiation of (7b) with respect to p yields $\nu'(p) = \sigma^{-1} r_x' k_x'$. Application of Lemma 1 yields:

Lemma 3: The growth rate of consumption is decreasing in p iff the education sector is relatively labor intensive ($k_x > k_y$).

If $k_x > k_y$, an increase in p decreases the net rental on capital, making investment in physical capital less attractive. More of good X is then allocated to current consumption rather than investment.

B. Output Supplies and Existence of the Balanced Growth Path

A balanced growth path is one in which C, H, and K all grow at constant rates. Lemma 3 established that in order for consumption to grow at a constant rate, the relative price of sector Y output must be a constant. From Lemma 2, the relative price in the balanced growth path, denoted p^*, is determined by the intertemporal no arbitrage condition and the requirement that the factor returns be equalized. In this section we complete our characterization of the balanced growth path by analyzing the conditions for a constant growth rate for human and physical capital.

Define $c = C/H$ and $k \equiv K/H$. We will establish the following result:

<u>Proposition 1</u>: If a balance growth path exists, it exhibits a common growth rate ν^* for consumption and the two factors. If the economy starts from the initial endowment k^*, then the balanced growth path will solve the first order conditions (3a) - (3g) of the dynamic optimization problem (P1) if $w_y - \eta > \nu^*$.

In order to characterize the balanced growth path and prove proposition 1, it will first be useful to derive comparative statics results for the effects of p and k on the sectoral outputs. The full employment condition for capital requires that $S_{hx}k_x(p) + (1-S_{hx})k_y(p) = k$, or $S_{hx} = (k-k_y(p))/(k_x(p)-k_y(p))$. Scaling the outputs of each sector by the stock of human capital, we can express outputs as $x(p,k) - X/H = S_{hx}(p,k)f(k_x(p))$ and $y(p,k) - Y/H = (1-S_{hx}(p,k))g(k_y(p))$. The following lemma summarizes the comparative statics effects of changes in prices and endowments on normalized outputs.

<u>Lemma 4</u>: Output supplies can be characterized as:

$$\frac{\partial x}{\partial p} < 0, \quad \frac{\partial y}{\partial p} > 0, \quad \frac{\partial x}{\partial k} = \frac{f(k_x)}{(k_x - k_y)}, \quad \frac{\partial y}{\partial k} = - \frac{g(k_y)}{(k_x - k_y)}.$$

The two factor model with constant returns to scale has a concave transformation schedule, which yields the normal output supply responses. The effects of endowments on output supplies are the well known Rybczynski Theorem from international trade theory: an increase in the aggregate capital/labor ratio results in a more than proportional expansion of the output of the capital intensive good and a decrease in the output of the labor intensive good.

Letting ν_i denote the growth rate of factor i, equations (1) and (2) can be rewritten as

$$\nu_H = \dot{H} / H = y(p,k) - \eta \tag{8a}$$

$$\nu_K = \dot{K} / K = k^{-1}[x(p,k) - c] - \delta. \tag{8b}$$

By definition, p and ν_i are constant along balanced growth paths. From (8), it is clear that constant growth rates for the capital stocks require c and k to be constant. This implies that a balanced growth path is one in which C, H, and K all grow at the common rate ν^*.[8] The values of c and k consistent with balanced growth can now be determined by solving (8) using $p = p^*$ and $\nu_H = \nu_K = \nu_C^*$.

We now establish that solutions c^* and k^* to (8) will exist if $w_y - \eta > \nu^*$. First, note that the aggregate factor proportion in the balanced growth equilibrium, k^*, is determined by the requirement that human capital grow at rate ν^*. In order for a balanced growth path to exist, we must have a solution of (8a) with $k \in [k_x, k_y]$. From Lemma 4, the right hand side of (8a) is linear in k. Since $S_{hx} = (k - k_x)/(k_y - k_x)$, we have $y(p, k_x(p)) = 0$ and

$y(p,k_y(p)) = g(k_y)$. Evaluating (8a) at these endpoints yields $\nu_H(k_x) = -\eta$ and $\nu_H(k_y) = g(k_y) - \eta > w_y - \eta$. Therefore, $w_y - \eta > \nu^* > 0$ is sufficient for the existence of a unique k^* consistent with balanced growth.

Given the solution for k^* from (8a), (8b) yields the normalized consumption in the balanced growth equilibrium, c^*. The condition $w_y - \eta > \nu^*$ is also sufficient to ensure $c^* > 0$. To establish this, note that normalized national income can be written as $x + py = w_x + r_x k$. Substituting for x and y from (8) and using the equalization of factor returns across sectors yields $c = (r_x - \delta - \nu_K) + p(w_y - \eta - \nu_H)$. Since $r_x - \delta = w_y - \eta$ in the steady state (from (5)), $w_y - \eta > \nu^*$ guarantees $c^* > 0$.

These results have established the existence of a balanced growth path that solves the conditions (3a)-(3e) of P1. To complete the proof of Proposition 1, it remains to show that this solution also satisfies the transversality conditions (3f) and (3g). From (3d), μ will grow at rate $\rho + \delta - r_x$ on the balanced growth path. Since K is growing at ν^*, the transversality condition (3f) will be satisfied if $r_x - \delta > \nu^*$. From above, this condition is satisfied by a balanced growth path with $c^* > 0$. A similar argument establishes that this condition also ensures that (3g) will be satisfied, which completes the proof of Proposition 1. Note that by substituting from the definition of ν_C in (7b), the condition for existence of a balanced growth path can be rewritten as $(r_x - \delta) > \rho/(1-\sigma)$. This condition is more likely to be satisfied the greater is σ, which corresponds to a lower intertemporal elasticity of substitution.

IV. Transitional Dynamics

In this section we analyze the transitional dynamics of the system in the neighborhood of the balanced growth path, and investigate the role played by the relative factor intensities of the two sectors in determining the dynamics of the system.

It is shown in the Appendix that this system can be analyzed using the normalized variables (i.e., $c = C/H$, $k = K/H$) and the relative price p. This result is shown by noting that due to the assumptions on preferences and technology, the problem (P1) can be equivalently formulated as

$$H_0^{1-\sigma} v(k_0) = \frac{\max}{c, S_{hx}, S_{kx}, k} \quad H_0^{1-\sigma} \int_0^\infty \frac{c(t)^{1-\sigma}}{1-\sigma} \, e^{-\Delta(t)} \, dt \qquad (P2)$$

where $\Delta(t) = \rho t - (1-\sigma)[\int_0^t ((1-S_{hx})g(k_y)-\eta)ds]$ and the transition rule for k is $\dot{k} = [S_{hx}f(k_x)-c-(1-S_{hx})g(k_y)k+(\eta-\delta)k]$. Expressing the problem in the form (P2) yields two insights about the optimization problem. First, it is shown in the Appendix that by making use of a variant of the Uzawa transformation, (P2) can be solved as an optimal control problem with a single state variable k, and this system has a saddle path which converges to steady state values k^* and c^*. Furthermore, it can be shown that the transversality condition for this transformed problem implies that $\dot{p} = 0$ along the saddle path. Second, since $V(K,H) = H^{1-\sigma}v(k)$, the function $V(K,H)$ is homogeneous of degree $1-\sigma$. This result will be useful in the analysis of the transitional dynamics below.

Utilizing (7) and (8), the dynamics of consumption and factor supplies can be expressed as

$$\dot{c}/c = \dot{C}/C - \dot{H}/H = \nu_C(p) - y(p,k) + \eta \qquad (9)$$

$$k'/k = \dot{K}/K - \dot{H}/H = k^{-1}[x(p,k)-c] - y(p,k) + \eta - \delta. \qquad (10)$$

Equations (5), (9), and (10) describe the evolution of the state variables p, c, and k.

The results of Lemmas 1-4 can now be used to analyze the transitional dynamics of the system. The linearized dynamic system is given by

$$\begin{bmatrix} \dot{p}/p \\ \dot{c}/c \\ \dot{k}/k \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} p - \bar{p} \\ c - \bar{c} \\ k - \bar{k} \end{bmatrix} \qquad (11)$$

where $\quad a_{11} = r'_x k'_x - w'_y k'_y$

$a_{21} = -\partial y/\partial p + \nu'$

$a_{23} = -\partial y/\partial k$

$a_{31} = k^{-1} \partial x/\partial p - \partial y/\partial p < 0$

$a_{32} = -k^{-1} < 0$

$a_{33} = -k^{-2} (x-c) + k^{-1} \partial x/\partial k - \partial y/\partial k.$

The lemmas above establish that both the price adjustment and quantity adjustment processes depend on which sector is physical capital intensive, so we shall consider each case separately.

Case 1: $k_x < k_y$

Lemmas 1 and 4 establish that $a_{11} > 0$, and a_{33}, $a_{23} < 0$ in this case. As noted above, the price adjustment process depends only on p and is unstable in this case , so p must jump immediately to the steady-state level. The determinant

of the 2x2 submatrix of (11) corresponding to the (c,k) dynamics is negative, and the matrix will have roots that are real and of opposite sign.

The system will have a saddle path as illustrated in Figure 2. The $\dot{c} = 0$ locus will be vertical, since there is a unique k (given p) at which C and H grow at the same rate. In this case an increase in the capital/labor ratio reduces the relative output of good x ($a_{33} < 0$), which is a stabilizing force because it reduces the accumulation of capital relative to labor (holding c constant). The $\dot{k} = 0$ locus must then be downward sloping, since an increase in k must be accompanied by a reduction in c to keep the two factors accumulating at the same rate. Examination of Figure 2 establishes that the saddle path must be upward-sloping. Starting from an initial $k_0 < k^*$, we must have $\nu_K > \nu_H$ and $\nu_C > \nu_H$ during the transition to the balanced growth path. Since the price remains constant at p^* throughout the transition, the growth rate of consumption will be constant at the steady state level ν^* from (7). It is shown in the Appendix that this path is the one that satisfies the transversality condition to (P2).

The only remaining question is to rank the growth rate of capital relative to that of consumption during the transition to the balanced growth path. From (8b), the slope of the locus of values in Figure 2 along which ν_K is constant will be downward-sloping for this case, since (x-c)/k is decreasing in k and c. Therefore, ν_K must be falling as k rises along the saddle path. Since $\nu_K = \nu^*$ in the balanced growth path, $\nu_K > \nu_C = \nu^*$ along the transition to the balanced growth path.

It should be noted that in deriving these results, we have assumed that both sectors are in operation along the optimal path. Since $p = p^*$ along the optimal path, it is straightforward to derive the range of initial values k_0

for which this assumption is satisfied. If $k_0 \in [k_x(p^*), k_y(p^*)]$, then the initial stock of capital is consistent with full employment factors and the economy will converge to k^* with both sectors in operation. If $k_0 < k_x(p^*)$, it is not possible to fully employ labor at p^* with both sectors operating. The wage rate must then fall further below $w_x(p^*)$ to fully employ labor, which will require that only sector X output be produced. Only X output will be produced until the aggregate k ratio rises to $k_x(p^*)$, at which point the economy will evolve as illustrated in Figure 2. Similarly, if $k_0 > k_y(p^*)$, no accumulation of capital will take place until the aggregate capital/labor ratio falls to $k_y(p^*)$.

Case 2: $k_x > k_y$

In this case, Lemmas 2 and 3 yield $a_{11} < 0$, $a_{21} < 0$, $a_{23} > 0$, and $a_{33} > 0$. As noted in Lemma 2, the price adjustment process is stable for this case. However, the dynamics of c and k (at fixed prices) are not stable, since an increase in k raises the output of x and reduces the output of y, leading to a further increase in k. Adjustments of the price along the optimal path will be necessary if the system is to be stable.

As a result of the restrictions noted above for this case, the determinant of the premultiplied matrix in (11) will be negative and the dynamic system will have either one negative root or three negative roots. One of the roots will equal a_{11}, which is negative. The two remaining roots will equal $[a_{33} \pm (a_{33}^2 + 4a_{32}a_{23})^{1/2}]/2$. Since $a_{33} > 0$ and $a_{23}a_{32} < 0$, these roots must both have positive real parts. Therefore, the whole system will again have a saddle path.

In the previous case it was possible to analyze the transitional

dynamics in (c,k) space because p was constant along the optimal path. In the present case, p will be changing along the saddle path, as illustrated in the three-dimensional representation in Figure 3a. However, it is possible to analyze the dynamics in (c,k) space making use of the fact that the value function is homogeneous of degree $1-\sigma$. Since the costate variables are the derivatives of the value function with respect to the respective state variables, the costate variables are homogeneous of degree $-\sigma$. Therefore, $p = \lambda/\mu$ is a function of k alone. Also, p(k) will be non-decreasing in k from the concavity of the value function.

Substituting for $dp = p'(k)dk$ in the second two rows of (11), the dynamic system can be simplified to

$$\begin{pmatrix} \dot{c}/c \\ \dot{k}/k \end{pmatrix} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} c - \bar{c} \\ k - \bar{k} \end{pmatrix} \tag{12}$$

$$b_{12} = -\frac{\partial y}{\partial k} + \left(\nu'(p) - \frac{\partial y}{\partial p} \right) p'(k)$$

$$b_{21} = -1/k < 0$$

$$b_{22} = \left(\frac{c-x}{k^2} + \frac{\partial x}{\partial k} \frac{1}{k} - \frac{\partial y}{\partial k} \right) + \left(\frac{1}{k} \frac{\partial x}{\partial p} - \frac{\partial y}{\partial p} \right) p'(k).$$

The term b_{12} captures two conflicting forces. An increase in k reduces the output of y (at constant prices) since $\partial y/\partial k < 0$ in this case, which tends to raise c. On the other hand, the rising p associated with an increase in k raises the growth rate of consumption ($\nu_C' > 0$ for $k_x > k_y$) and raises output of y, both of which tend to reduce c. However, the fact that the system has a saddle path can be used to guarantee that the latter effect will dominate, so that $b_{12} < 0$. To see this note that the determinant of the system is $-b_{12}b_{21}$,

which is equal to the product of the characteristic roots of the matrix in
(12). Since the system has a saddle path, the roots must be of opposite sign,
which requires $b_{21} < 0$.

The term b_{22} also involves conflicting forces. An increase in k will
increase the relative output of good x when $k_x > k_y$. However, the rising p
will reduce the relative output of good x. The fact that the system has a
saddle path is not sufficient to determine which of these effects dominates.
We will thus have two cases to consider. If $b_{22} < 0$, the stabilizing effect
of p on relative outputs dominates the Rybczynski effect and the $\dot{k} = 0$ locus
will be downward-sloping. The $\dot{c} = 0$ locus will be vertical. Since the sign
pattern of the matrix governing dynamics in (c,k) space is identical to that
for the previous case $(k_x < k_y)$, the saddle path will be upward-sloping as
illustrated in Figure 2. If $b_{22} > 0$, the Rybczynski effect on outputs
dominates the effect of a rising price and the $\dot{k} = 0$ locus will be upward
sloping. The phase diagram for this case is illustrated in Figure 3b, which
shows that the saddle path must be upward sloping for this case as well.
Therefore, in either case we have that if $k_0 < k^*$, $\nu_K > \nu_H$ along the
transition path and $\nu_C > \nu_H$. These two results are similar to those obtained
for the case $k_x < k_y$.

The difference from the previous case where $k_x < k_y$ is that the rising k
must cause p to rise. From Lemma 4 we know that the growth rate of
consumption is a decreasing function of p when $k_x > k_y$. The growth rate of
consumption will be falling along the optimal path, so $\nu_C > \nu^*$ during the
transition to balanced growth. There are conflicting forces operating on the
growth rates of both types of capital during the transition. The increase in
k tends to raise the growth rate of physical capital and reduce the growth

rate of human capital when $k_x > k_y$, but the increasing price of human capital

has the opposite effect. Therefore, it will not in general be possible to

rank ν_H and ν_K relative to ν^* for the transition. Similarly, it will not in

general be possible to rank ν_K relative to ν_C for this case. The slope of the

locus along which $\nu_K = \nu_C$ is $dc/dk = - (b_{22}-b_{12})/b_{21}$. For the case $b_{22} > 0$ in

Figure 3b, this locus must be steeper than the $k = 0$ locus, and its slope

cannot be ranked in general relative to the saddle path. Similarly, for the

case $b_{22} < 0$ (as in Figure 2), the locus may be upward-sloping so its slope

cannot be ranked relative to the saddle path. In the case $k_x > k_y$, we obtain

a wider range of possible rankings of growth rates because of the conflicting

role of price effects and factor accumulation effects on the outputs of the

two goods.[9]

Proposition 2: The balanced growth equilibrium is saddle path stable

regardless of the factor intensities. Starting from $k_0 < k^*$, then in the

neighborhood of the balanced growth path we have physical capital and

consumption both growing more rapidly than labor. If $k_x < k_y$, then $\nu_K >$

$\nu_C = \nu^* > \nu_H$. If $k_x > k_y$, then $\nu_C > \nu^*$.

V. Applications and Conclusions

We conclude by illustrating how the model can be used to analyze the

effects of changes in parameters and public policy on the balanced growth

path.

A. Changes in ρ and δ

A decrease in the discount rate will make investment in both physical and human capital more attractive. Since both types of investment are affected equally, there is no effect on the intertemporal no arbitrage condition (5) and the relative price of human capital in the balanced growth path will be unaffected by the change in the discount rate. Since p* is unaffected and $\nu_C = (r_x - \rho - \delta)/\sigma$, a decrease in the discount rate must raise the growth rate in the balanced growth path.

Equations (9) and (10) can be used to calculate the effect of this increase in the balanced growth rate on c and k. Differentiating (9) yields $dk/d\nu^* = (\partial y/\partial k)^{-1}$. An increase in the growth rate requires an increase in the output of the educational sector, since all Y sector output is used for factor accumulation. An increase in Y output requires an increase in k iff $k_y > k_x$ from Lemma 4. Totally differentiating (8b) and utilizing $dk/d\nu^*$ yields

$$\frac{dc}{d\nu*} = \left[-k\frac{\partial y}{\partial k} + \frac{\partial x}{\partial k} - \left(\frac{x-c}{k}\right)\right] / \left(\frac{\partial y}{\partial k}\right) < 0. \tag{13}$$

An increase in the growth rate requires an increase in (x-c)/k in order to increase the rate of capital accumulation. If $k_y > k_x$, then $(\partial x/\partial k) < 0$ and $(\partial y/\partial k) > 0$, and the bracketed expression in (9) must be negative. An increase in the growth rate raises k and reduces x, so the only way that (x-c)/k can increase is for consumption to fall. If $k_x > k_y$, then by Lemma 4 we have $\partial x/\partial k > x/k > 0$ and $\partial y/\partial k < 0$, and the bracketed expression must be positive. In this case, $dk/d\nu^* < 0$ and output of x must again fall. The percentage reduction in output will be greater than the percentage reduction in k, however, so that x/k falls. In order to have the growth rate increase,

c/k must fall by more than x/k, which requires a reduction in c.

We next examine the effect of an increase in the rate of depreciation on physical capital, δ. An increase in δ reduces the attractiveness of physical capital, which must alter the relative price of sector Y output on the balanced growth path. Differentiating (5) yields

$$\frac{dp}{d\delta} = \alpha^{-1} \tag{14a}$$

$$\frac{dv}{d\delta} = (\alpha\sigma)^{-1} w_y \frac{dk_y}{dp} \frac{dp}{d\delta} < 0 \tag{14b}$$

$$\text{where } \alpha = r_x \frac{dk_x}{dp} - w_y \frac{dk_y}{dp}.$$

Utilizing $r_x < 0$, $w_y > 0$, and Lemma 1, sign α = sign $(dp/d\delta)$ = sign $(k_y - k_x)$. An increase in the depreciation rate makes physical capital more expensive, and will raise the relative price of the capital-intensive good. Note also that the growth rate effect is larger when σ is low.

The effect of an increase in δ on k is obtained by differentiating (8a),

$$\frac{\partial y}{\partial k} \frac{dk}{d\delta} = \left(\frac{\partial v}{\partial \delta} - \frac{\partial y}{\partial p} \frac{\partial p}{\partial \delta} \right).$$

In the case where $k_y > k_x$, the right hand side of the expression is negative from (14) and Lemma 4. It follows that $dk/d\delta < 0$ in this case. If $k_x > k_y$, the right hand side is ambiguous because the negative effect of δ on growth conflicts with the effect of falling p on Y output. We will have $dk/d\delta < 0$ in this case only if the latter effect dominates.

The comparative statics results of this section can be summarized as:

<u>Proposition 3</u>:

(a) An decrease in the discount factor ρ will raise the balanced growth rate and reduce the level of consumption per effective labor unit on the balanced growth path. The capital-labor ratio will fall iff $k_x < k_y$.

(b) An increase in δ will reduce the growth rate. The relative price of y will rise iff $k_x < k_y$. If $k_x < k_y$, then $dk/d\delta < 0$.

B. Factor Taxation

This model can also be used to compare the effects of capital and labor taxation on the balanced growth path. Capital taxation is assumed to consist of a tax at rate τ_K on the earnings from all capital. Labor taxation is modelled as a tax at rate τ_L on earnings from human capital in the goods sector. The asymmetry between capital and labor taxation is that it is assumed that labor in the education sector is untaxed, since the foregone earnings of labor are not taxed during the process of human capital accumulation.

The tax revenues, T, collected from factor taxation are assumed to be used to purchase goods which enter separably in the household utility function and thus do not affect household preferences over private goods. The objective function for the household then remains to solve (P1) for a given tax policy, subject to (2) and

$$\dot{K} = S_{hx}Hf(k_x) - T - \delta K - C \qquad (15)$$

where $T \equiv r_x(1-\tau_K)K + w_x(1-\tau_L)S_{hx}H$. Forming the current value Hamiltonian for this problem and taking the first order conditions with respect to S_{kx} and S_{hx}, respectively, yields

$$r_x(k_x) = pr_y(k_y) \qquad (16a)$$

$$w_x(k_x)(1-\tau_L) = pw_y(k_y) \tag{16b}$$

where r_i (w_i) is the marginal product of physical (human) capital in sector i and p is the relative price of human capital as before. Conditions (16) require equalization of the net of tax return to factors. Since capital is taxed at the same rate in both sectors, this results in the equalization of the pre-tax returns to capital as in (4a) for the case without factor taxation. The marginal product of labor will be higher in the goods sector because of the assumption that labor in human capital accumulation is untaxed. Defining μ and λ to be the costate variables associated with K and H respectively, the necessary conditions also require

$$\dot{\mu} = \mu(\rho+\delta) - \mu S_{kx}f' - \lambda(1-S_{kx})g' \tag{17a}$$

$$\dot{\lambda} = \lambda(\rho+\eta) - \mu S_{hx}(f-k_xf') - \lambda(1-S_{hx})(g-k_yg'). \tag{17b}$$

Since $p = \lambda/\mu$, (17) can be combined to yield

$$\frac{\dot{p}}{p} = r_x(k_x(p))(1-\tau_K) - w_y(k_y(p)) + \eta - \delta. \tag{18}$$

Equation (18) is the intertemporal no-arbitrage condition for the case with factor taxation, which requires that the net of tax returns to the two factors be equalized.

Equations (16) and (18) can be solved for the values of k_x, k_y, and p consistent with the balanced growth path ($\dot{p}=0$) as was done in Section III for the case without factor taxation. Totally differentiating (16) and (18) yields the following result:

Lemma 5:

(a) The effects of an increase in a tax on capital income are:

$$\frac{\partial k_x}{\partial \tau_K} = \frac{pr_x(w_y + k_y r_y)}{r_x' D} < 0, \quad \frac{\partial k_y}{\partial \tau_K} < 0, \quad \frac{\partial p}{\partial \tau_K} \gtreqless 0 \text{ as } k_y \gtreqless k_x(1 - \tau_K).$$

(b) The effects of a tax on labor income in the goods sector are:

$$\frac{\partial k_x}{\partial \tau_L} = \frac{-w_x r_y k_y}{r_x' D} > 0, \quad \frac{\partial k_y}{\partial \tau_L} < 0, \quad \frac{\partial p}{\partial \tau_L} < 0$$

where $D \equiv r_y(k_y(k_x(1-\tau_L) + k_y(1-\tau_K)) + w_y(k_y + p(1-\tau_K))) > 0$.

A tax on capital income raises the cost of physical capital, leading to substitution of human capital for physical capital in both sectors. The price of educational services will rise as a result of the tax on capital iff education is capital intensive relative to goods production. The effect of a capital tax is similar to that obtained above for the effect of an increase in the depreciation rate, except that the definition of capital intensity relevant for determining the direction of the price change involves a comparison of k_y with $k_x(1-\tau_L)$. This comparison is equivalent to a comparison of the ratio of capital to labor costs in the two sectors.[10]

Since the tax on labor income is limited to the goods sector, it raises the cost of labor relative to capital in the goods sector, but reduces it in the education sector. The tax must reduce the relative price of education output, since the tax falls only on the goods sector. In contrast to the capital tax, the impact on price does not depend on the factor intensity ranking of the two sectors.

Lemma 5 can be used to calculate the effects of factor taxation on the rate of growth. Differentiating the Hamiltonian with respect to C yields the condition $C^{-\sigma} - \mu = 0$. Differentiating this with respect to time and substituting from (16a) yields the growth rate of consumption to be

$$\nu_C = \sigma^{-1} \left[r_x(k_x(\tau_K, \tau_L))(1-\tau_K) - (\rho+\delta) \right]. \qquad (19)$$

Factor taxes affect growth through their impact on the net return to capital
in the goods sector. Since labor taxation raises k_x, an increase in the rate
of labor taxation must reduce the net return to capital and the rate of growth
on the balanced growth path, $d\nu/d\tau_L = -(1-\tau_K)w_x r_y k_y/\sigma D < 0$. An increase in
the rate of capital taxation reduces the net return to capital at a given k_x,
but it also reduces k_x. Differentiation of (19) establishes that $d\nu/d\tau_K =$
$- r_x k_y(r_y k_x(1-\tau_L)+w_y)/\sigma D < 0$, so that the former effect must dominate. Thus,
both types of factor taxes reduce the rate of growth, and, as a consequence,
retard human capital accumulation. The latter contrasts with Heckman (1976)
in which a general capital tax, under a partial equilibrium framework, reduces
the after-tax real rate of interest and thus encourages human capital
accumulation.

The capital tax involves no allocative distortion at a point in time,
but creates an intertemporal distortion in (17a) by reducing the future return
on capital investments. In contrast to Lucas (1990), where labor taxation
results only in a human wealth effect, the labor tax in present work has two
effects. There is both an intertemporal distortion, which retards human
capital accumulation, and an allocative distortion at a point in time, which
enourages education relative to employment in the X sector. The relative
effect of the two taxes on the growth rate can be seen by comparing the effect
of an increase in the two taxes such that $d\tau_K/(1-\tau_K) = d\tau_L/(1-\tau_L)$, which
reduces the net of tax return on each factor at the same rate. Comparing the
above results for the effects of the taxes establishes that the capital tax
has a larger impact on the growth rate, since $(1-\tau_K)d\nu/d\tau_K < (1-\tau_L)d\nu/d\tau_L$.
The capital tax has a more negative impact on growth because of its broader

coverage.[11] In summary, we have

Proposition 4: Both capital taxation and labor taxation must reduce the rate
of economic growth. Labor taxation must raise p, and raises (lowers) the
capital labor ratio in sector X (Y). Capital taxation raises p iff $k_x < k_y$,
and will raise the capital-labor ratio in both sectors in all cases.

C. Extensions

   This main objective of this paper has been to develop a general two
sector model of endogenous growth with human capital and physical capital
accumulation, where labor and capital are used in both production sectors. We
have shown how the use of the intertemporal no-arbitrage condition yields a
simplified analysis of the balanced growth path, and have shown how this
technique can be applied to analyze effects of parameter changes.

   We conclude with some brief comments on how the techniques in this paper
can be applied to other extensions of the model. The first issue concerns the
welfare analysis of taxation. We have briefly analyzed the effect of factor
taxes on the rate of growth, but a more complete analysis would compare the
effects of the two taxes on welfare given a government revenue objective. In
addition, the model can be used to analyze other forms of taxes, such as
sectoral output taxes. A second extension is to incorporate leisure time into
the analysis. If leisure time enters the utility function, households must
also choose the amount of human capital to be allocated to leisure activities.
Such an extension does not affect the intertemporal no-arbitrage condition or
sectoral zero profit conditions, but requires adjustments in the output
equations to distinguish between the stock of human capital and the stock that
is employed in production. Finally, the model could also be extended to allow

for externalities from the stock of capital of the type analyzed by Romer (1986).

1.   Mulligan and Sala-i-Martin (1992) examine transitional dynamics in a two sector model of endogenous growth, but their analysis is limited to simulation. Caballe and Santos (1991) examine transitional dynamics in the Lucas two sector model, with the accumulation of human capital requiring only labor inputs.

2.   There is an interesting comparison between the instability of the quantity adjustment described in this paper with that obtained in the two sector growth model of Uzawa (1961, 1963) where only capital accumulation is endogenous. In the Uzawa model, instability may arise when the investment good is capital intensive relative to the consumption good. This is analogous to our case in which the education sector is labor intensive, since each factor is used intensively in its own production. In contrast to Uzawa, however, we obtain saddle path stability for either factor intensity ranking because of the role played by the price adjustment process. It should also be noted that Shell, Sidrauski, and Stiglitz (1969) have considered the role of capital gains in the Uzawa model where a constant fraction of income is save.

3.   This formulation allows for the perpetual evolution of human capital, which accords with the observation that ancestor's human capital skill levels can affect the descendant's through home education and intergenerational knowledge transfers. This point is discussed by Ehrlich and Liu (1991).

4.   Inclusion of capital in the education sector accounts for the observed production similarity between physical and human capital inputs (see Becker (1971), Ben-Porath (1967), and Schultz (1961)).

5.   For a discussion on this restriction, see Rebelo (1991).

6.   In deriving the necessity of transversality conditions (3f) and (3g) for this problem, Benveniste and Scheinkman impose an assumption that the objective function u is non-negative. For $\sigma > 1$, this conditions will not be met by (P1). However, the purpose of this assumption is to assure the positivity of the state vector and of the costate variables. Assumptions which accomplish this, and are met by (P1), can be used to show that (3f) and (3g) are necessary for this problem.

7.   We assume throughout this section that both goods and education are produced. This result can be modified in ways that will be noted below to include the possibility that initial conditions are such that only one sector output is produced at some points in the transition toward the balanced growth path.

8.   Caballe and Santos (1991) obtain a similar condition for the existence of a balanced growth path in the case where only labor is used in the production of human capital. Thus, allowing both factors to be used in both sectors does not alter the conclusions regarding the existence of a balanced growth path. However, it does play an important role in the transitional dynamics and the analysis of the effects of parameter changes on the balanced growth path, which will be considered below.

9.   As in the previous case, this analysis holds assuming that the non-negativity constraints on factor accumulation are not binding. This is equivalent to assuming that $k \in [k_y(p), k_x(p)]$ along the optimal path. This restriction cannot be simply illustrated in (c,k) space, since p (and thus the bounds of the feasible region) are changing along the path.

10.   Jones (1971) refers to this as the value factor intensity ranking of goods, while a comparison of k_x and k_y is referred to the physical factor intensity ranking. In cases where there are distortions in factor markets, these rankings will not necessarily coincide. Note that the rankings here differ only because of the labor tax, since the capital tax is the same across sectors.

11.   Clearly, this comparison of the effect of the two types of factor taxes takes no comparison of the revenue raised from the two types of taxes. In order to compare the welfare effects of the two types of taxes, it would be necessary to compare the effects on utility of labor and capital taxes that raise the same amount of revenue. Such an analysis is beyond the scope of this paper, and remains an area for future work.

REFERENCES

Barro, Robert. J. "Government Spending in a Simple Model of Endogenous
     Growth." Journal of Political Economy 98 (October 1990): S103-S125.

Becker, Gary S. Human Capital and the Personal Distribution of Income: An
     Analytical Approach. Ann Arbor: MI, Institute of Public
     Administration, University of Michigan, 1967.

Ben-Porath, Yoram. "The Production of Human Capital and the Life Cycle of
     Earnings." Journal of Political Economy 75 (August 1967): 352-365.

Benveniste, Lawrence M. and Jose A. Scheinkman, "Duality Theory for Dynamic
     Optimization Models of Economics: The Continuous Time Case." Journal
     of Economic Theory, 27, (1982): 1-19.

Caballé, Jordi and Manuel S. Santos. "On Endogenous Growth with Physical and
     Human Capital." Working Paper, Barcelona: Spain, Universitat Autonoma
     de Barcelona, 1991.

Ehrlich, Isaac and Francis Lui. "Intergenerational Trade, Longevity, and
     Economic Growth." Journal of Political Economy 99 (October 1991):
     1029-1059.

Hall, Robert. "Intertemporal Substitution in Consumption." Journal of
     Political Economy 96 (April 1988): 339-357.

Heckman, James J. "A Life-Cycle Model of Earnings, Learning, and
     Consumption." Journal of Political Economy 84 (August 1976): S11-S44.

Jones, Larry E. and Rodolfo Manuelli. "A Convex Model of Equilibrium Growth:
     Theory and Policy Implications." Journal of Political Economy 98
     (October 1990): 1008-1038.

Jones, Ronald W. "Distortions in Factor Markets and the General Equilibrium
     Model of Production." Journal of Political Economy 73 (May/June 1971):
     437-59.

Jones, Ronald W. "The Structure of Simple General Equilibrium Models."
     Journal of Political Economy 73 (December 1965): 557-572.

Judd, Kenneth L. "The Welfare Cost of Factor Taxation in a Perfect-Foresight
     Model." Journal of Political Economy 95 (August 1987): 675-709.

King, Robert G. and Sergio Rebelo. "Public Policy and Economic Growth:
     Developing Neoclassical Implications." Journal of Political Economy 98
     (October 1990): S126-S150.

Lucas, Robert E., Jr. "On the Mechanics of Economic Development." Journal of
     Monetary Economics 22 (August 1988): 3-42.

Lucas, Robert E., Jr. "Supply-side Economics: An Analytical Review." <u>Oxford Economic Papers</u> 42 (April 1990): 293-316.

Mulligan, Casey B. and Xavier Sala-i-Martin. "Transitional Pyamamics in Two-Sector Models of Endogenous Growth." Working Paper, Cambridge: MA, NBER, 1992.

Rebelo, Sergio. "Long-run Policy Analysis and Long-run Growth." <u>Journal of Political Economy</u> 99 (April 1991): 500-521.

Romer, Paul. "Increasing Returns and Long-Run Growth." <u>Journal of Political Economy</u> 94 (October 1986): 1002-1037.

Schultz, Theodore W. "Investment in Human Capital." <u>American Economic Review</u> 51 (March 1961): 1-17.

Shell, Karl, Miguel Sidrauski, and Joseph Stiglitz. "Capital Gains, Income, and Saving," <u>Review of Economic Studies</u> 36 (January 1969): 15-26.

Uzawa, Hirofumi. "On a Two-sector Model of Economic Growth, I." <u>Review of Economic Studies</u> 14 (January 1961): 40-47.

Uzawa, Hirofumi. "On a Two-sector Model of Economic Growth, II." <u>Review of Economic Studies</u> 19 (January 1963): 105-118.

Uzawa, Hirofumi. "Optimal Technical Change in an Aggregative Model of Economic Growth." <u>International Economic Review</u> 6 (February 1965): 18-31.

FIGURE 1a

Determination of Equilibrium Effective
Capital—Labor Ratios
$$(k_x > k_y)$$
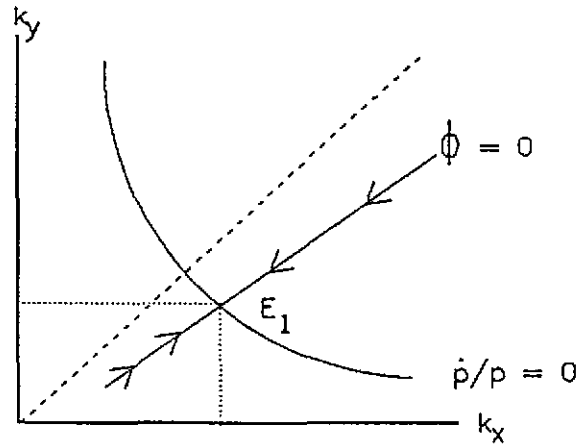
FIGURE 1b

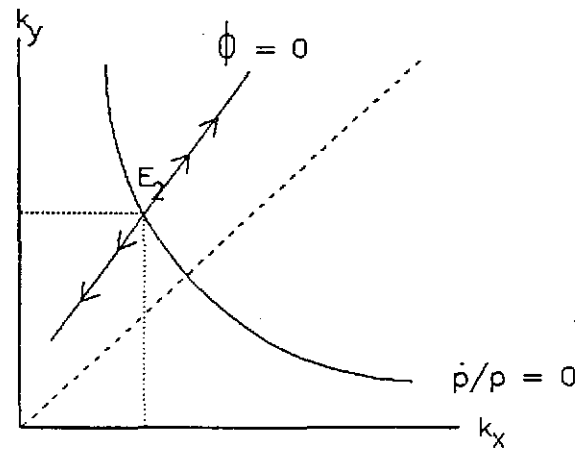Determination of Equilibrium Effective
Capital—Labor Ratios
$$(k_x < k_y)$$

FIGURE 2

Saddle−Path Stability: $k_x < k_y$
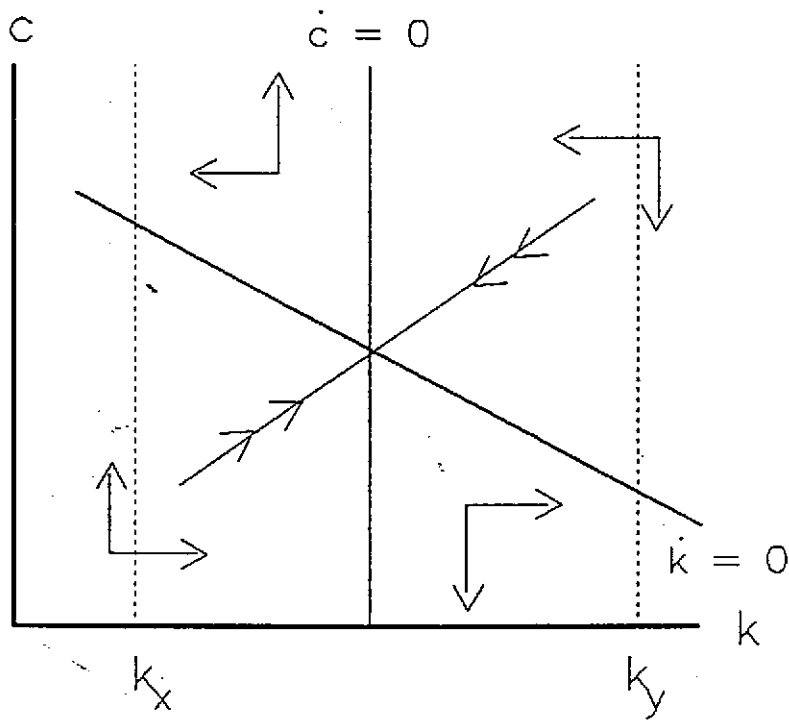### (Projection on the c−k Space)
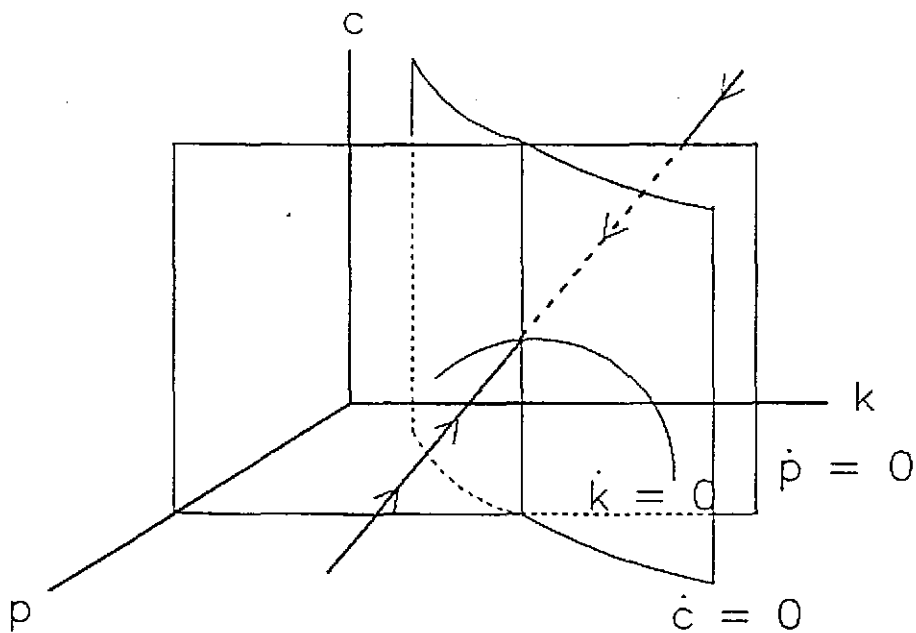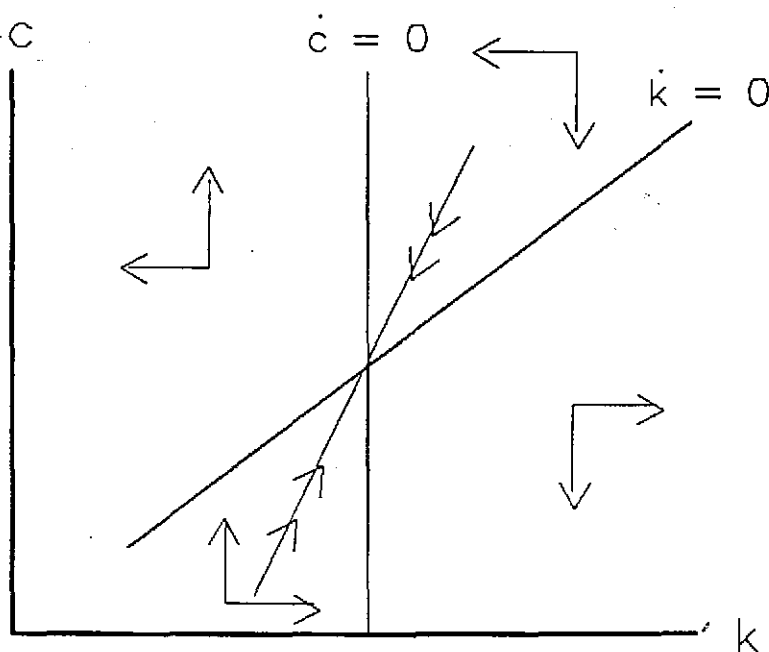
FIGURE 3a
Saddle–Path Stability: $k_x > k_y$

FIGURE 3b
Projection on c–k Space: $k_x > k_y$

APPENDIX

The purpose of this Appendix is to present a brief analysis of the transformed problem (P2), which expresses variable in per unit human capital terms. By presenting the problem using this transformation, we illustrate the behavior of the ratio of the capital stocks, k, along the optimal path. Proposition 1 established the existence of a balanced growth path along which both factors grow at the same rate, which implies $\dot{k} = 0$. Our objective in this section is to analyze the transversality condition for k, and to show that $\dot{k} \to 0$ implies $\dot{p} \to 0$. This allows us to analyze the transitional dynamics of the untransformed problem (P1) using the relative price p. In addition, we illustrate that the problem (P1) is homogeneous of degree $1-\sigma$, and establish the linkage between the costate variables of the two problems.

Since the discount factor in (P2) depends on the control variables, it is convenient to solve for the optimal policies as a function of Δ, using the fact that $d\Delta = \Gamma(t)dt$, $\Gamma(t) = \rho - (1-\sigma)\nu_H(t)$. This transformation turns the problem into one in which standard techniques from optimal control theory can be applied. The problem (P2) can then be restated as

$$H_0^{1-\sigma}v(k_0) = \frac{\max}{c,S_{hx},S_{kx},k} \quad H_o^{1-\sigma} \int_0^\infty \frac{c(t)^{1-\sigma}}{(1-\sigma)\Gamma} e^{-\Delta} d\Delta \qquad (A.1)$$

subject to $dk/d\Delta = [S_{hx}f(k_x)-c-(1-S_{hx})g(k_y)k+(\eta-\delta)k]/\Gamma$.

To solve this problem, form the Hamiltonian

$$J(k,S_{hx},S_{kx},c,\theta) = \left[(c^{1-\sigma}/(1-\sigma)) + \theta\left(S_{hx}f(k_x) - c - (1-S_{hx})g(k_y)k + (\eta-\delta)k\right)\right]/\Gamma. \qquad (A.2)$$

For $\sigma > 1$, this problem will be concave in the control variables and will satisfy the conditions of Bennenisle and Scheinkman (1982). The necessary

conditions are

$$c^{-\sigma} - \theta = 0 \tag{A.3a}$$

$$\theta(r_x + k\, r_y) - J^*(1-\sigma)\, r_y = 0 \tag{A.3b}$$

$$\theta(w_x + k\, w_y) - J^*(1-\sigma)w_y = 0 \tag{A.3c}$$

$$\frac{d\theta}{d\Delta} = \theta - \frac{\partial J}{\partial k} = \theta - \frac{\theta}{\Gamma}\left[S_{kx}\, r_x + (\eta - \delta) - (1 - S_{hx})\, g(k_y) - (1 - s_{kx})\, r_y\, k\right]$$

$$\tag{A.3d}$$

$$- \frac{J^*(1-\sigma)}{\Gamma}\, (1 - s_{kx}) r_y$$

$$\lim_{\Delta \to \infty} k\theta e^{-\Delta} = 0 \tag{A.3e}$$

where J^* is the value of the Hamiltonian (A.2) at its maximum.

The costate variable θ has the interpretation of being the value of a unit of physical capital per unit of human capital. It is useful in interpreting (A.3) to relate this variable to the variables μ, λ, and p used in analyzing P1. Since the costate variables are the derivatives of the value functions with respect to the respective state variables, we have $\theta = v'(k)$, $\mu = V_k$, and $\lambda = V_H$. Differentiation of the identity $H^{1-\sigma}v(k) = V(kH, H)$ yields the relations:

$$\theta = H^{\sigma}\mu \tag{A.4a}$$

$$H^{\sigma}\lambda = (-\sigma)v(k) - k\,\theta \tag{A.4b}$$

$$p = \frac{(1-\sigma)\, v(k)}{\theta} - k. \tag{A.4c}$$

Using (A.4), it can be seen that (A.3a-c) will be equivalent to (3a-c) if $v(k) = J^*$. This equality will hold since $v(k) = \int_0^\infty J^* e^{-\Delta}\, d\Delta - \int_0^\infty \theta\, \frac{dk}{d\Delta} e^{-\Delta}$. Using $v'(k) = \theta$ and integrating by parts yields the desired equality.

Equation (A.3d) can be rewritten using (A.3b) as:

$$\frac{1}{\theta}\ \frac{\mathrm{d}\theta}{\mathrm{dt}} = \frac{\Gamma}{\theta}\ \frac{\mathrm{d}\theta}{\mathrm{d}\Delta} = \rho + \delta - r_x + \sigma\left(\left(1 - S_{hx}\right) g\left(ky\right) - \eta\right) \qquad\qquad (A.5)$$

(A.5) implies that in the steady state where $\frac{\mathrm{d}\theta}{\mathrm{d}\Delta} = 0$, we have $\nu_H = (1 - S_{hx})$

$g(k_y) - \eta = (\rho + \delta - r_x)/\sigma$, which is identical to the steady state growth rates

obtained using (7) and (8a) in the text.

Finally, note that since $\dot{k} \to 0$ and $\dot{\theta} \to 0$ in the limit, then

differentiation of (A.4c) establishes $\dot{p} \to 0$.

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