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Intratemporal Welfare and the Optimal Depletion
of Exhaustible Resources

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1. Introduction

The optimal depletion literature has often given normative significance to particular depletion plans for exhaustible resources. Optimal depletion studies typically rely on models that use the objectives of a "social planner," or equivalently, a social welfare function in the objective functional. The most commonly used welfare function has been the unweighted aggregate sum of consumer's surplus (i.e., the area under the demand curve). Using this particular measure of social welfare and abstracting from problems caused by the existence of externalities, uncertainty, or extraction costs, Hotelling (1931) established an equivalence between the depletion paths resulting from an unfettered competitive system and that produced by a social planner.

The public finance literature, on the other hand, has explored the limitations of aggregate consumers' surplus as a measure of social welfare in static models and has isolated several important assumptions necessary for its use as an unbiased measure. If the utility derived from consumption by different individuals is not weighted differently, aggregate consumer surplus provides a useful measure of social welfare. If there is reason to doubt the optimality of the underlying distribution of income that generates the observed demand for the good, however, the assumption that aggregate consumer surplus appropriately captures social welfare is more difficult to justify.^{1/}

This paper examines the implications of relaxing that assumption in the case of the optimal depletion of exhaustible resources. The social welfare function is allowed to apply different weights to the consumer surplus of different consumers. Although some investigation of the problems of intertemporal welfare optimization has been undertaken,^{2/} the effect of the "intratemporal" welfare function (i.e., the social welfare function at any point in time) on the optimal depletion path has not been adequately explored.

Two results are derived in this paper for the more general case where within period welfare and intertemporal welfare are jointly treated using a generalized individualistic social welfare function (GISWF). First, it is shown that the use of differential welfare weighting results in the finding that a perfectly competitive system no longer necessarily yields a depletion path equivalent to the socially optimal path. Instead, the optimal depletion path is slower (faster) than the competitive path depending on whether individuals with larger welfare weights have less (more) elastic demands for the resource than those with smaller weights.

Second, welfare weighting is shown to raise problems with second order conditions. If rents are more heavily distributed to those with low weights, it is possible that the social welfare function becomes convex in consumption of the resource, implying that the Euler equations fail the sufficiency criterion. Both of these results are demonstrated for the two-person case.

The generalized solution of the optimal depletion problem in a multi-consumer framework is presented first in Section 2. Problems raised

for second-order conditions by welfare weighting are then addressed in Section 3. In Section 4, the two-person case is extended to include the possibility of rent transfers between consumers. It is shown that the inclusion of rent redistribution increases the likelihood of generating a convex social welfare function. Finally, implications of the results are presented in Section 5.

2.1 The Model

Using a GISWF to determine an optimal depletion path generates additional restrictions on the family of admissible social welfare functions. This section develops the optimal conditions for a depletion path when the welfare function is expanded to include n consumers.

The nature and implications of the GISWF are discussed first. Once this welfare function is defined, the Euler equations are derived to characterize the optimal depletion path of an exhaustible resource. The special case with welfare measured by aggregate consumer surplus is then compared to a weighted social welfare function in both the general case and two-person cases. Finally, implications of incorporating rents from the resource that are distributed unequally are examined.

Let the GISWF be defined by

$$(1) \quad W(\underline{h}(t)) = W(U_1(h_1(t)), \dots, U_n(h_n(t))),$$

where $W(\underline{h}(t))$ is defined as the social welfare derived from the $n \times 1$ vector of individual levels of consumption of the resource, \underline{h} , in period t . The function $U_i(h_i)$ is the utility person i derives from the consumption of h_i units of the resource, where $h(t)$ is the sum of all h_i in period t .

Equation 1 assumes that the utility derived by each individual can be ultimately described as a reduced form function of resource use. To simplify the analysis, it is assumed that all consumption and production decisions occur in the presence of perfect information, and that the production level of the resource is the only policy tool the controller possesses. The quantity h_i is the share of h which is embedded in the final consumption bundle of person i in the period specified.

The optimal depletion path is one which solves the problem:

$$(2) \quad \text{Max}_{\underline{h}} \int_0^T e^{-rt} W(\underline{h}(t)) dt + W(T)$$

subject to $\dot{R} = -h(t)$,

$$\int_0^T h(t) dt \leq R(0), \text{ and}$$

$$h_i(t) \geq 0 \text{ for all } i \text{ and } t,$$

where R is the stock of the resource available, r is the social rate of time preference, and $W(T)$ is a final function specifying the level of welfare possible without the resource. The problem in equation 2 determines a level of resource use each period that maximizes the discounted present value of welfare, taking into account the allocation of the given resource across the n consumers. The equations solve this level for each period in time subject to the restrictions that (a), the resource stock is decreased by the level of current consumption, (b), cumulative consumption over time cannot exceed the available stock, and (c), consumption each period is nonnegative.

The Euler equation can be derived with the aid of the present value Hamiltonian,

$$(3) \quad H(t) = e^{-rt} W(\underline{h}(t)) - \lambda(t)h(t).$$

Since $\partial H/\partial R = -\dot{\lambda}(t) = 0$, λ is not a function of time. Therefore, the equation

$$(4) \quad \partial H/\partial h = -e^{-rt} W'(\underline{h}(t)) - \lambda = 0,$$

defines the Euler equation. Since $e^{-rt} W'(\underline{h}(t))$ is equal to a constant, λ , the Euler equation specifies a path satisfying

$$(5') \quad e^{-r(t-s)} W'(\underline{h}(t)) = W'(h(s)),$$

or equivalently:

$$(5) \quad \frac{W''(h)}{W'(h)} = r.$$

In contrast, the original socially optimal path derived by Hotelling can be written:

$$(6) \quad \frac{\dot{P}(t)}{P(t)} = r,$$

where P is the price of the resource net of extraction costs (assumed constant). This latter rule, that the net price of the resource should rise at the rate of discount, is a special case of 5 when $W(\underline{h})$ is defined by aggregate consumer surplus because marginal social welfare is equal to the price along the demand curve. Equations 5 and 6 are not equivalent if the marginal social welfare function is not proportional to the aggregate demand curve for all prices at which some consumption occurs. To

demonstrate the conditions under which 5 and 6 differ, it is useful to examine the two person case.

2.2. The Two Person Case

For two persons, the welfare function, $W(\underline{h})$, can be written as

$$(7) \quad W(\underline{h}) = W(U_A(h_A), U_B(h_B))$$

where $U_A(h_A)$ and $U_B(h_B)$ are the utilities derived from individual consumption of the resource by A and B respectively. Equation 5' implies that

$$(8) \quad \frac{\partial W}{\partial U_A} \cdot \frac{dU_A(t)}{dh_A(t)} \cdot \frac{dh_A(t)}{dh(t)} + \frac{\partial W}{\partial U_B} \cdot \frac{dU_B(t)}{dh_B(t)} \cdot \frac{dh_B(t)}{dh(t)} =$$

$$e^{-r(s-t)} \frac{\partial W}{\partial U_A} \cdot \frac{dU_A(s)}{dh_A(s)} \cdot \frac{dh_A(s)}{dh(s)} + \frac{\partial W}{\partial U_B} \cdot \frac{dU_B(s)}{dh_B(s)} \cdot \frac{dh_B(s)}{dh(s)}$$

Let social welfare in any period be approximated by

$$(9) \quad W(\underline{h}(t)) = w_A \int_0^{h_A} D_A(z) dz + w_B \int_0^{h_B} D_B(z) dz$$

where $D_A(h_A(t))$ is the demand curve for consumer A,

$D_B(h_B(t))$ is the demand curve for consumer B, and

w_A and w_B are the weights assigned to A and B respectively.

Equation 9 is a social welfare function in which welfare is approximated by the weighted sum of consumer surpluses. For this case, optimality would require

$$(10') \quad e^{-rt} [w_A s(t) + w_B (1-s(t))] P(t) = \\ [w_A s(0) + w_B (1-s(0))] P(0),$$

or, to use the alternative form in equation 5 and rearranging terms,

$$(10) \quad \frac{W'(h)}{W(h)} = \frac{\dot{P}}{P} - \frac{\dot{s} [w_B - w_A]}{P [w_A s + w_B (1-s)]} = r$$

where $s(t)$ is defined by dh_A/dh and $(1-s(t))$ is defined by dh_B/dh .

Using equation 10, it is clear that differences between the competitive and socially optimal price paths develop if w_A does not equal w_B or if \dot{s} is not equal to zero. If $w_A < w_B$ (as will be assumed throughout this section), then differences depend on the sign of \dot{s} (note that the denominator, $w_A s + (1-s)w_B$, is greater than zero). If $\dot{s} < 0$, implying that the share of an additional unit of h consumed by person A declines as the price of h increases, the optimal price path is slower than the competitive path. The opposite is true if $\dot{s} > 0$, implying that consumers with low weights have increasingly more inelastic demands than those with high weights as the price rises.^{3/}

Equation 10 demonstrates that Hotelling's competitive pricing rule is socially optimal if there is no welfare weighting, or if demand functions by all households are proportional. Furthermore, proportionality must hold for all prices, i.e., Hotelling's competitive path will not be optimal if some households have zero consumption at high prices (while others are still consuming) and yet have positive consumption at lower prices.

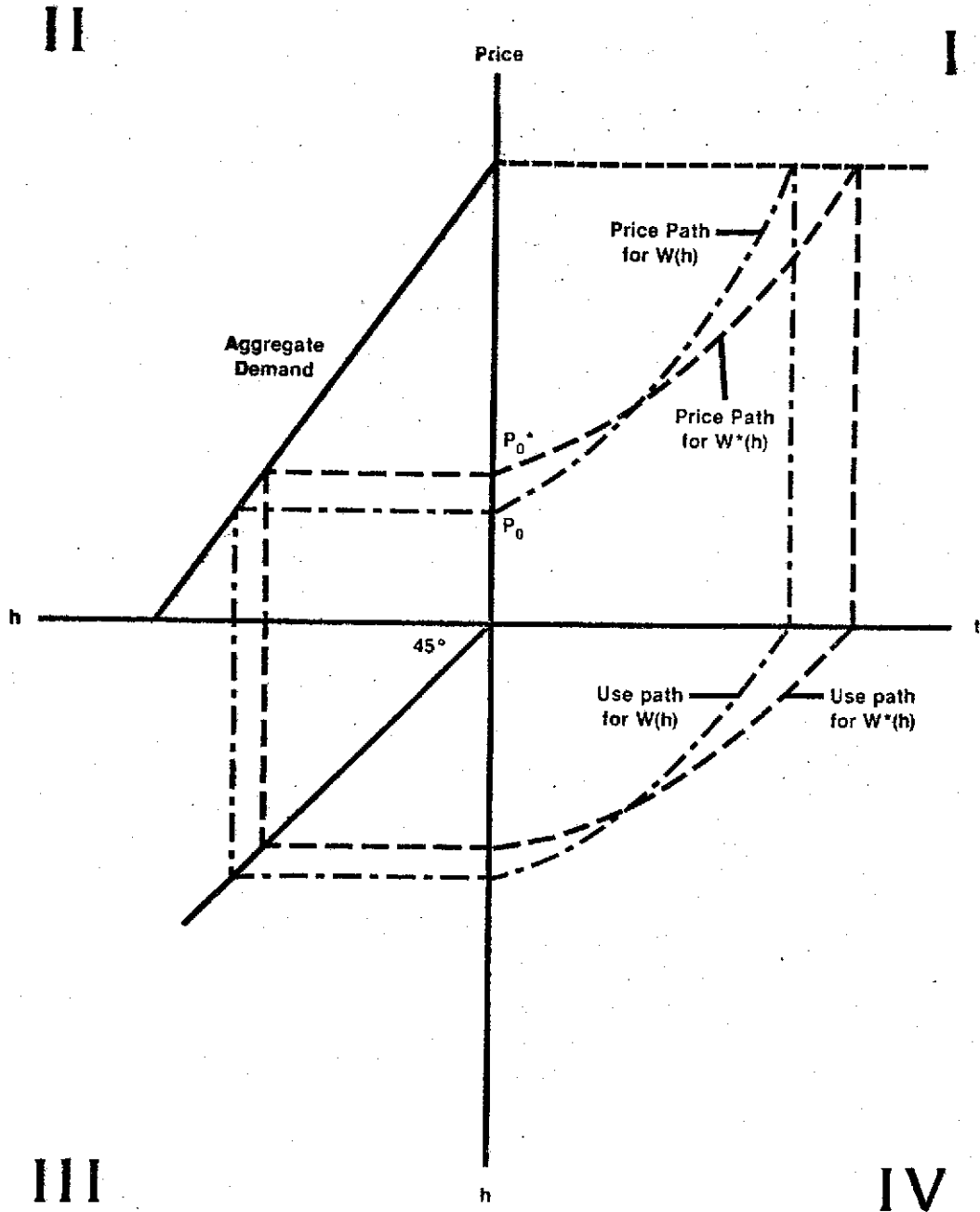
Differences in the price paths can be shown graphically using a four-quadrant diagram. Assuming that two alternative functions $W(h)$ and $W^*(h)$ exist and are concave in h , the optimal depletion paths implied by the two functions can be compared.

Let $W(h)$ be aggregate consumer surplus, while $W^*(h)$ applies differential welfare weights. Assuming that $\delta < 0$ (which again implies that B 's share of consumption of the marginal unit increases as the price of h rises over time), equation 10 implies that the socially optimal price path, P^* , should grow at a slower rate than the competitive path, P .

The effect on consumption is shown in Figure 1. The two price paths are shown in the first quadrant, and the market demand (which allocates the resource) is shown in the second quadrant. The third quadrant maps consumption from the price/quantity dimension to the quantity/time dimension, shown in quadrant four. This last quadrant maps the aggregate consumption path over time for each price path.

The optimal price and consumption paths shown in the figure are determined using two equations. First, the slope of each price path satisfies the rule derived in equation 10 that marginal social welfare rises at the discount rate. Second, endpoints (initial and final prices) are fixed using the Euler equation and the resource constraint. This constraint is represented graphically by the area bounded by the function in quadrant IV and the axes and represents cumulative use over time. This cumulative use cannot exceed the total stock of the resource which is available.

Figure 1: Comparison of Price and Consumption Trajectories for two Alternative Welfare Functions



Given the price paths, endpoints can be established by choosing an initial price, applying the Euler equation, and comparing the total stock consumed conditional on that starting point to the total stock available. If the aggregate use over time exceeds (is less than) the total stock, the starting price is raised (lowered) so that consumption in all periods is decreased (increased). This procedure is followed until the starting price and the Euler equation exactly exhaust the resource stock.^{4/}

Because the price path for the socially optimal case is slower than the competitive path, the consumption path for $W^*(h)$ supplies less of the resource than would result from $W(h)$ in the initial periods. In later periods, however, the quantity supplied in the socially optimal case exceeds that supplied in the competitive case.

The implication of Figure 1 and equation 10 is that a unique optimal path exists for each welfare function. The "optimal" depletion path generated by perfect competition is socially optimal if and only if the marginal social welfare function $W'(h)$ is proportional to the aggregate demand curve. Thus, the rule that the net resource price should rise at the rate of discount is not necessarily an optimal rule unless aggregate consumer surplus is an acceptable welfare measure.

3. Second Order Considerations

To this point, it has been assumed that the welfare functions are concave in resource consumption. The use of unequal welfare weights, however, can result in convex welfare functions. To simplify the notation, the time subscript will be suppressed while examining concavity conditions.

Concavity requires that $W'(\underline{h}) \geq 0$ and $W''(\underline{h}) \leq 0$. Assuming that there is no satiation, it is not necessary to look at the case where $W''(\underline{h})=0$. The problem, therefore, is to show conditions where $W'(\underline{h}) > 0$ and $W''(\underline{h}) < 0$.

The first order condition for a maximum requires that

$$(11) \quad d[W(U_1, \dots, U_n)]/dh = 0.$$

Carrying out the differentiation yields:

$$(12) \quad (\partial W/\partial U_1)(dU_1/dh_1)(dh_1/dh) \\ + \dots + (\partial W/\partial U_n)(dU_n/dh_n)(dh_n/dh) = 0.$$

The expression $\partial W/\partial U_i$ is greater than zero for all i as long as the welfare function does not assign negative values to the utility of some persons. The second term in each product, dU_i/dh_i , is positive as long as utility increases monotonically with additional consumption of the resource. The expression dh_i/dh is positive for all i if persons increase their own use of the resource as a unit of h is added. If h is allocated by a price mechanism, this condition simply requires that the derived demand curves for all consumers slope downward. (Equation 12 would also be satisfied as long as some individuals have downward sloping demand curves and are able to outweigh those with upward sloping demands).

The sufficiency condition for the Euler equation requires $W''(\underline{h}) < 0$. To simplify the notation, let $w_i = \partial W/\partial U_i$ be a welfare weight for consumer i , let $MU_i = dU_i/dh_i$, and let $dh_i/dh = s_i(h)$. Equation 12 can be rewritten as

$$(12') \quad W'(\underline{h}) = \sum_i (w_i \cdot MU_i \cdot s_i(h)) \geq 0.$$

To further simplify the problem, let w_i be independent of the magnitude of h . For the Euler equation in equation 5 to be optimal, it is necessary and sufficient for 12' to be satisfied and

$$(13) \quad W''(\underline{h}) = \sum_i w_i \cdot [(dMU_i/dh) \cdot s_i(h) + MU_i \cdot (ds_i(h)/dh)] < 0.$$

Define the expressions $[(dMU_i/dh) \cdot s_i(h)]$ as J , and $[MU_i \cdot (ds_i(h)/dh)]$ as K . Since $s_i(h)$ cannot be less than zero (that would imply that it would be possible to use negative amounts of h), diminishing marginal utility of consumption is a sufficient condition to make J negative for all i .

The sign of K is ambiguous. The value of MU_i is greater than zero by assumption. The second term in K , $ds_i(h)/dh$ (the rate at which person i increases consumption of h as total consumption increases), however, is likely to differ depending on the individual's preference for goods which use h intensively versus goods which do not use h intensively. If $s_i(h)$ does not change with h (i.e., consumers continuously purchase the same share of each new unit of h), then $ds_i/dh=0$, and $K=0$. Furthermore, if $ds_i/dh < 0$ for persons with high welfare weights, or high MU relative to those with $ds_i/dh > 0$, then K is less than zero and $W''(\underline{h}) < 0$. Finally, even if $K > 0$, as long as $|\sum w_i J| > |\sum w_i K|$, $W(\underline{h})$ will still be concave.

For the special case where social welfare in a period is defined by aggregate consumer surplus, the Euler equations are optimal. Defining

$D(h)$ as the aggregate demand for h by consumers, the welfare function is given by

$$(14) \quad W(\underline{h}) = \int_0^{\underline{h}} D(z) dz.$$

Optimality is assured as long as $D(h) > 0$ and $D'(h) < 0$ for all levels of h . Given nonnegative prices and downward sloping demand curves, the path described by Hotelling's rule is optimal.

4. Second Order Conditions, Rent, and the Two Person Case

The social welfare function is also affected by the inclusion of unequal welfare weights for different consumers if rents from the sale of the resource are distributed disproportionately to different groups. In the unweighted case, the distribution of rents had no impact on the solution because the social value of a dollar of rent was the same for all households. When differences in weights are allowed, however, a dollar of rent received by household A may have a different social value than a dollar of rent received by household B. As a result, unless each household receives a lump sum transfer equal to the rent derived from its expenditure on the resource, the social welfare function must also take into account the fact that consumption also implies a transfer of wealth through scarcity rents.

The implications of using welfare weights on the optimal depletion path can be shown by modifying the social welfare function defined in the previous example to incorporate rent. Rent in this case is defined as the

difference between the price charged for h and the marginal extraction cost (assumed to be zero in this example), multiplied by the quantity consumed by A and B. If the rent from person B's consumption, $P(t)h_B(t)$, is distributed to A, then the measurement of welfare defined in 9 will be incorrect.

To include the possibility that rent will be redistributed, the welfare weighted function in 9 can be redefined as

$$(15) \quad W^*(\underline{h}) = w_A \int_0^{h_A} D_A(z) dz + w_B \int_0^{h_B} D_B(z) dz - c(w_B - w_A) h_B P,$$

where the parameter c ($0 \leq c \leq 1$) is the fraction of B's rent transferred to A by virtue of A's larger ownership share of the resource. If $c=0$, then the problem collapses to the problem presented above in 9. The final term in 15 represents the new weight placed on the rent derived from B's consumption when part of it is distributed to A. (Person A is assumed to own a larger share of the resource than B, although the nature of the problem is unaffected if roles are reversed). The term $(w_B - w_A)$ is the weight applied to the rent from B's consumption of h and $D_B(h_B)h_B(t)$ is the magnitude of that rent at time t .^{5/}

Differentiating 15 with respect to h yields

$$(16) \quad W^*(\underline{h}) = (w_A \cdot s + w_B \cdot (1-s))P - c(w_B - w_A)[(dP/dh)h_B + (1-s)P].$$

The effect of adding rent to the model on the optimal use path can be seen in the final term. If $c=0$ or $w_A = w_B$, then the final term vanishes and the optimal path is that specified in 10. As long as the final term is

nonzero, however, the optimal path is altered by taking into account the distribution of rents. (Note that $W^* \geq 0$ since $(1-s)$ and c are both in the interval $[0,1]$).

Equation 16 can be differentiated with respect to h to yield the necessary second order condition:

$$(17) \quad (w_A s + w_B (1-s)) \frac{dP}{dh} + 2c(w_B - w_A)(1-s) dp/dh < \\ (w_B - w_A) [-c(d^2P/dh^2)h_B + P(1+c)(ds/dh)].$$

If $c=0$ or $w_A=w_B$, then the right hand side term vanishes and downward sloping demands for A and B and positive welfare weights are sufficient. With $c > 0$ and $w_B > w_A$, it is possible that 17 will not be satisfied.

Concavity in 17 depends on the relative slopes of the demands by A and B for h . As long as $dP/dh < 0$ (i.e., demand slopes downward), the left hand side of 17 is less than zero. Since $(w_B - w_A) > 0$ by assumption, a sufficient condition for concavity is for the term in brackets on the right hand side to be positive.

The first term in the bracket depends on the curvature of the aggregate demand curve. If the demand curve is approximately linear, then the first term is insignificant. Furthermore, the sign may be positive or negative. The final term, however, is negative when consumer B has a more inelastic demand curve for h than A: because the demand curve for B is more inelastic than the aggregate demand curve, more of an additional unit of h will go to A than to B. As a result, the share of B's marginal consumption increases as h falls. In order to violate concavity

conditions, however, it is necessary for this final term to be relatively large (in absolute value) so as to offset the other terms in the expression.

Hotelling's rule, therefore, can break down if there is a large discrepancy in welfare weights, most of the rent accrues to persons with low weights, and persons with high weights have relatively inelastic demand curves. Consider the extreme case with a perfectly inelastic demand curve for B and all rent transferred to person A. If the price of h increases, an amount equal to the price change times the quantity demanded (which is fixed) becomes producer surplus (rent) rather than consumer surplus. If producer surplus receives a lower weight than consumer surplus for B, welfare in the period would be decreased at an increasing rate. In other words, $W(h)$ will not be concave in h if an increasing proportion of consumer surplus plus rent goes to persons with low weights, and persons with high weights have relatively inelastic demand curves. If $W(h)$ is not concave in h, then Hotelling's rule fails the sufficiency criterion.

5. Implications

The results of this analysis have interesting implications. Consumer groups, such as the Citizens/Labor Energy Coalition and the Energy Action Educational Foundation (1981) have argued for maintaining price controls on natural gas to keep the price below market levels on the grounds that such a policy helps the poor. Assuming that the social welfare function is concave, the results of this exercise seem to imply the opposite: as

long as the poor have relatively less elastic demands for natural gas and are assigned higher welfare weights, the best way to help the poor may be by encouraging higher current prices.

Intuitively, low income consumers, it is assumed, are hurt more than the wealthy for each unit of decreased consumption of natural gas. A recommendation of high prices assumes that current prices are below the social value of the marginal unit of natural gas. Furthermore, the solution must also consider consumption by future generations of poor. By going to the "socially optimal" path -- with higher near-term prices and less near-term consumption -- today's poor suffer so that future generations of poor are better off. A policy of price controls below market levels, on the other hand, harms intertemporal social welfare by encouraging excessive consumption today (in addition to discouraging conservation and new technologies), at the cost of future consumption.

This prescription, however, must be tempered by other factors. First, the choice of a discount rate is critical. If a discount rate approaching infinity is used, future consumption does not contribute significantly to social welfare, so a policy of low prices would maximize the present value of social welfare by helping the current generation of consumers. Second, costs of adjustment need to be taken into account. Rather than pursuing bang-bang optimal control (i.e., moving from current prices to optimal prices immediately), phased deregulation programs may be preferred. Because it is costly and time consuming to change from gas use to other energy forms, it could be argued that extensive social costs would occur with an immediate switch to a "socially optimal" path.

Finally, the distribution of rents from the sale of natural gas could also affect the optimal path. If rents are heavily distributed to upper income households away from low income households, consumer groups may argue that a policy of holding down current prices maximizes social welfare, because the social welfare function may be convex.

Footnotes

1. The debate surrounding the use of aggregate consumer surplus can be found in Harberger (1971), Willig (1976), Burns (1973), Boadway (1974), and Silberberg (1972).
2. Solow (1974), for example, experimented with the concept of a "Rawlsian" intertemporal social welfare function. His approach relied on using the discount rate to reflect society's evaluation of the relative marginal utilities of different generations.
3. In this analysis, it is assumed that the relative demand functions of different consumers are stationary over time.
4. Given the construction of this model, with zero extraction costs, a finite choke price, and known reserves, the stock of reserves can be shown to be depleted completely in finite time. (See Dasgupta and Heal (1974)).
5. The weight $w_B - w_A$ is used rather than w_A to avoid counting the rent from B's consumption twice.

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