Federal Reserve Bank of Dallas

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Globalization Institute Working Paper 389 **May 2020**

Research Department

https://doi.org/10.24149/gwp389

A Matter of Perspective: Mapping Linear Rational Expectations Models into Finite-Order VAR Form*

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May 10, 2020

Abstract

This paper considers the characterization of the reduced-form solution of a large class of linear rational expectations models. I show that under certain conditions, if a solution exists and is unique, it can be cast in finite-order VAR form. I also investigate the conditions for the VAR form to be stationary with a well-defined residual variance-covariance matrix in equilibrium, for the shocks to be recoverable, and for local identification of the structural parameters for estimation from the sample likelihood. An application to the workhorse New Keynesian model with accompanying Matlab codes illustrates the practical use of the finite-order VAR representation. In particular, I argue that the identification of monetary policy shocks based on structural VARs can be more closely aligned with theory using the finite-order VAR form of the model solution characterized in this paper.

JEL Classification: C32, C62, C63, E37.

Keywords: Linear Rational Expectations Models; Finite-Order Vector Autoregressive Representation; Sylvester Matrix Equation; New Keynesian Model; Monetary Policy Shocks.

* I would like to thank Nathan S. Balke, Christiane Baumeister, James B. Bullard, Menzie D. Chinn, Olivier Coibion, Mario J. Crucini, Morris A. Davis, Charles Engel, Jesús Fernández-Villaverde, Marc P. Giannoni, Yuriy Gorodnichenko, Andrés Giraldo, James D. Hamilton, Urban Jermann, John Keating, Lutz Kilian, Thomas A. Lubik, Rodolfo Manuelli, María Teresa Martínez-García, M. Hashem Pesaran, Michael Plante, Giorgio Primiceri, Erwan Quintin, Federico Ravenna, Michael Sposi, Mark A. Wynne, and many seminar and conference participants for helpful suggestions and comments. All codes for this paper are available using the following link: https://bit.ly/2ZAHcvy. I acknowledge the excellent research assistance provided by Valerie Grossman. All remaining errors are mine alone. The views expressed in this paper are my own and do not necessarily reflect the views of the Federal Reserve Bank of Dallas or the Federal Reserve System.

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1 Introduction

The solution of linear or linearized rational expectations (LRE) models is an important part of modern macroeconomics. Blanchard and Kahn (1980) established the conditions under which a solution to an LRE model exists and is unique (see also the related contributions and extensions of, among others, Broze et al. (1985), Broze et al. (1990), King and Watson (1998), Uhlig (1999), and Klein (2000)). The unique solution of an LRE model, when one exists, can be represented generically in linear state-space form as seen, e.g., in Fernández-Villaverde et al. (2007) and Morris (2016).

Structural (finite-order) VAR models in the spirit of Sims (1980) provide a useful and widely popular framework to investigate and organize the evidence on a set of observable variables. However, mapping the identification restrictions arising from theory onto an estimated structural VAR is not without its problems. Satisfying the 'poor man's invertibility condition' of Fernández-Villaverde et al. (2007) suffices to ensure that, whenever the number of structural shocks is equal to the number of observable endogenous variables, the unique linear state-space representation of the LRE model solution takes a companion VAR(∞) form. However, unless a finite-order VAR representation describes exactly the solution, using it introduces a truncation error when the unique LRE solution in fact is described by an infinite-order VAR (Inoue and Kilian (2002)).

Morris (2016) has shown that any linear state-space framework can be cast in VARMA form and, in some cases, can be reduced to a finite-order VAR form (see also the related illustrations in Morris (2017)).[1] Ravenna (2007) studying the same question posed by this paper on the mapping of the reduced-form solution of an LRE model finds a finite-order VARMA representation. The key contribution of this paper is to show that imposing the cross-equation restrictions derived from the LRE model along the lines of Ravenna (2007) suffices to prove by characterization that the unique solution of an LRE model, when one exists, can be cast into a companion VAR specification of finite order.

The paper works out this result for a large class of LRE models whose endogenous variables can be described with a canonical first-order expectational difference system of equations and where the forcing variables behave as a VARMA$(1,1)$ process.[2] Working with this, the paper shows that the reduced-form solution of a purely-forward looking LRE model inherits the dynamics of the forcing variables and, therefore, the endogenous variables display the dynamics of a VARMA$(1,1)$ process. The paper goes further showing that, in some cases, the dynamics of the endogenous variables are simply those of a VAR(1) process even when the forcing variables follow a VARMA$(1,1)$. Moreover, if the forcing variables behave as a VAR(1) process, so do the endogenous variables.

The paper also finds that decoupling the backward- and forward-looking parts of the LRE model along the lines of Binder and Pesaran (1995) and Binder and Pesaran (1997) is useful to characterize the reduced-form solution with lagged endogenous variables. In that situation, the dynamics of the endogenous variables are shown to follow a VARMA$(2,1)$ process. The endogenous variable dynamics are those of a VAR(2)

[1] Franchi and Paruolo (2015). explores general conditions that ensure any linear state-space representation can be re-written as a finite-order VAR. Their related results apply to any type of linear state-space specification. Instead, by making use of the cross-equation restrictions implied by the LRE model, this paper simplifies the derivation of the finite-order VAR representation of the LRE solution and the conditions required to guarantee its existence, uniqueness, and fundamentalness. Moreover, it also ties the companion finite-order VAR form directly to the deep structural parameters and relationships of the LRE model and facilitates the mapping of the implied cross-equation restrictions onto structural (finite-order) VARs for estimation and identification purposes.

[2] The canonical first-order system in (1) − (3) can be generalized to include LRE models with more than one lead and one lag of the endogenous variables in W_t and of the forcing variables in X_t, as explained in Broze et al. (1985), Broze et al. (1990).

whenever the forcing variables follow a VAR(1) but also in some cases when they follow a $\text{VARMA}(1,1)$.[3] The paper also shows that, whenever the number of shocks is equal to the number of endogenous variables, the finite-order VAR solution if it exists is a well-behaved multivariate process and satisfies the fundamentalness property of Hansen and Sargent (1980). That is, it is stationary, its variance-covariance matrix is symmetric and positive definite, and the corresponding shock innovations are recoverable.

Finally, the paper illustrates the practical use of the finite-order VAR solution with an illustration based on the workhorse New Keynesian model. In doing so, I contribute to the ongoing debate on the assumptions needed to correctly recover theoretically-consistent monetary policy shocks through structural VARs (see, e.g., Carlstrom et al. (2009)). In particular, I highlight some of the contradictory empirical inferences that can arise when using standard identification strategies (Cholesky, zero-restrictions, etc.) instead of the identification restrictions implied by theory.

The rest of the paper proceeds as follows: Section 2 describes the mapping of the LRE model solution into finite-order VAR form via companion quadratic and Sylvester matrix equations and the conditions under which a unique and well-behaved finite-order VAR solution exists. I also explore the canonical representation of the $\text{VARMA}(1,1)$ process for the forcing variables as a way to accommodate moving average terms into the framework. The procedure to characterize the finite-order VAR solution is easily cast in an algorithmic form and a collection of Matlab implementation codes is provided with the paper.[4] Section 3 applies the companion finite-order VAR structure to identify and explore the propagation of monetary policy shocks using a variant of the workhorse New Keynesian model. Section 4 then concludes.

2 Mapping LRE Models into Finite-Order VARs

A large class of LRE models can be cast into a canonical first-order expectational difference system of equations, featuring forward- and backward-looking dynamics. The first-order expectational difference equations capture the structural relationships between a set of $k \geq 1$ forcing variables $X_t = (x_{1t}, x_{2t}, ..., x_{kt})^T$ and $m \geq 1$ endogenous variables $W_t = (w_{1t}, w_{2t}, ..., w_{mt})^T$ as follows:

$$W_t = \Phi_1 W_{t-1} + \Phi_2 \mathbb{E}_t [W_{t+1}] + \Phi_3 X_t + \Phi_4 \mathbb{E}_t [X_{t+1}] . \tag{1}$$

The conforming matrices Φ_1 and Φ_2 are $m \times m$ square matrices, while Φ_3 and Φ_4 are $m \times k$ matrices. In principle, as is the case for most LRE models, the expectational difference system in (1) includes $m \geq k$ endogenous variables in W_t. However, the selection of m endogenous variables in W_t can be set to include exactly the same number of endogenous variables as of forcing variables (that is, $m = k$) by dropping some of the endogenous variables from W_t or by adding measurement error on some of them increasing the elements in X_t.

The LRE model may contain $p \geq 0$ other endogenous variables not in W_t all of which are collected in the vector $\widetilde{W}_t = (\widetilde{w}_{1t}, \widetilde{w}_{2t}, ..., \widetilde{w}_{pt})^T$. Then, the vector $\widetilde{W}_t$ can be expressed as a linear transformation of the

[3] It is worth noting that an LRE model whose solution can be cast with a linear state-space setup that includes lagged endogenous variables would have a VARMA representation according to Morris (2016), but the paper shows that the solution can still be represented in finite-order VAR form even in cases like that when imposing the cross-equation restrictions from theory.

[4] All codes for this paper are available using the following link: https://bit.ly/2ZAHcvy.

subset of endogenous variables in W_t and possibly of the vector of forcing variables X_t, i.e.,

$$\widetilde{W}_t = \Lambda_1 W_t + \Lambda_2 X_t, \tag{2}$$

with conforming $p \times m$ matrix Λ_1 and $p \times k$ matrix Λ_2. Hence, given the exogenous dynamics for X_t, solving for W_t and characterizing its endogenous dynamics suffices to completely describe all other endogenous variables $\widetilde{W}_t$ of the LRE model by means of (2).

The (first-order) specification for the vector of forcing variables X_t is given by a standard VARMA$(1,1)$ of the following form:

$$X_t = A_1 X_{t-1} + u_t + A_2 u_{t-1},\ u_t \equiv B\epsilon_t, \tag{3}$$

including k exogenous shock innovations in $\epsilon_t = (\epsilon_{1t}, \epsilon_{2t}, ..., \epsilon_{kt})^T \sim i.i.d.\,(0, \mathbf{I}_k)$ where $\mathbf{I}_k$ denotes the $k \times k$ identity matrix. The fact that there are as many driving processes as shock innovations (k) is often satisfied by construction. When the model has more forcing variables than shock innovations, each of the forcing variables can be expressed as a function of just k of them and, therefore, some can be replaced out so as to end up with just k forcing variables. The real matrices A_1, A_2, and B are $k \times k$ square matrices which satisfy the following assumptions:

Assumption 1 *A_1 has all its eigenvalues inside the unit circle ensuring the stationarity of the stochastic VARMA$(1,1)$ process for X_t in (3) and, accordingly, A_1 is invertible (has full rank, i.e., rank $(A_1) = k$). A_2 has all its eigenvalues inside the unit circle ensuring the invertibility of the stochastic VARMA$(1,1)$ process for X_t in (3) and, accordingly, A_2 is also invertible (has full rank, i.e., rank $(A_2) = k$).*

Assumption 2 *B has all its eigenvalues inside the unit circle and, accordingly, is invertible (has full rank, i.e., rank $(B) = k$). The corresponding variance-covariance matrix given by $\left(B^T B\right)$ is therefore symmetric, positive definite, and invertible.*

2.1 A Canonical Representation of the VARMA$(1,1)$ Process

The moving average representation of the stochastic process for the forcing variables X_t in (3) can be expressed as follows:

$$X_t = u_t + \sum\nolimits_{j=1}^{\infty} A_1^{j-1} (A_1 + A_2)\, u_{t-j}. \tag{4}$$

From here, I can derive the canonical representation of the VARMA$(1,1)$ process as suggested by de Jong and Penzer (2004). The moving average form makes it straightforward to define the minimum mean squared error predictor $X_{t|t-1} \equiv \mathbb{E}_{t-1}[X_t]$ as:

$$X_{t|t-1} = X_t - u_t = \sum\nolimits_{j=1}^{\infty} A_1^{j-1} (A_1 + A_2)\, u_{t-j}. \tag{5}$$

Thus, I can re-write the moving average representation in (4) shifted one period ahead as:

$$\begin{aligned} X_{t+1} &= u_{t+1} + \sum\nolimits_{j=1}^{\infty} A_1^{j-1} (A_1 + A_2)\, u_{t+1-j} \\ &= u_{t+1} + (A_1 + A_2)\, u_t + A_1 \sum\nolimits_{j=1}^{\infty} A_1^{j-1} (A_1 + A_2)\, u_{t-j} \\ &= u_{t+1} + (A_1 + A_2)\, u_t + A_1 X_{t|t-1}, \end{aligned} \tag{6}$$

where the last equality follows from (5). Hence, from (5) and (6), I obtain that the vector of forcing variables X_t can be described with a linear state-space system using the following state equation:

$$X_{t+1|t} = X_{t+1} - u_{t+1} = (A_1 + A_2)\, u_t + A_1 X_{t|t-1}, \tag{7}$$

and the corresponding observation equation given by:

$$X_t = X_{t|t-1} + u_t. \tag{8}$$

Equations (7) and (8) can be used in place of the VARMA$(1,1)$ specification given by equation (3).

Given the structure of the VARMA$(1,1)$ for the forcing variables in (3), the structural relationships implied by the expectational difference system in (1) can be re-written as follows:

$$W_t = \Phi_1 W_{t-1} + \Phi_2 \mathbb{E}_t [W_{t+1}] + (\Phi_3 + \Phi_4 A_1)\, X_t + \Phi_4 A_2 u_t. \tag{9}$$

In the special case where the stochastic process for X_t does not have a moving average component (that is, when $A_2 = \mathbf{0}_k$, with $\mathbf{0}_k$ being the $k \times k$ matrix of zeros), then equation (1) does not include explicitly the vector of random disturbances given by $u_t \equiv B\epsilon_t$. In general, however, the canonical (first-order) LRE model given by (1) and (3) can be re-written now as follows:

$$W_t = \Phi_1 W_{t-1} + \Phi_2 \mathbb{E}_t [W_{t+1}] + (\Phi_3 + \Phi_4 A_1)\, X_{t|t-1} + (\Phi_3 + \Phi_4 (A_1 + A_2))\, B\epsilon_t, \tag{10}$$

$$X_{t+1|t} = A_1 X_{t|t-1} + (A_1 + A_2)\, B\epsilon_t. \tag{11}$$

This canonical representation implies that the redefined forcing variables $X_{t+1|t}$ has an autoregressive structure and that the vector of shock innovations ϵ_t appears both in (10) and (11).

If a solution to the canonical first-order LRE model given by (10) – (11) exists, then it can be written in linear state-space form as:

$$W_t = C X_{t|t-1} + \Theta W_{t-1} + D\epsilon_t, \tag{12}$$

together with the dynamics of $X_{t+1|t}$ given by (11). Here, C and D are real-valued $m \times k$ matrices and Θ is a real-valued $m \times m$ square matrix, all of them unknown. Equations (12) and (11) can be cast into the general linear state-space representation studied by Morris (2016).

2.2 Decoupling Backward- and Forward-Looking Terms

The canonical system in (10) – (11) can be decoupled in its backward- and forward-looking parts, along the lines of Binder and Pesaran (1995) and Binder and Pesaran (1997). Let us assume that the $m \times m$ square matrix Θ in (11) is used to transform the vector of endogenous variables W_t into $Z_t \equiv W_t - \Theta W_{t-1}$. Replacing the vector of endogenous variables with this transformation into the expectational difference system in (10) implies that:

$$Z_t + \Theta W_{t-1} = \Phi_1 W_{t-1} + \Phi_2 [\mathbb{E}_t (Z_{t+1}) + \Theta (Z_t + \Theta W_{t-1})] + (\Phi_3 + \Phi_4 A_1)\, X_{t|t-1} + (\Phi_3 + \Phi_4 (A_1 + A_2))\, B\epsilon_t, \tag{13}$$

which then becomes:

$$\left(\mathbf{I}_m - \Phi_2\Theta\right) Z_t = \Phi_2 \mathbb{E}_t \left(Z_{t+1}\right) + \left(\Phi_2\Theta^2 - \Theta + \Phi_1\right) W_{t-1} + \left(\Phi_3 + \Phi_4 A_1\right) X_{t|t-1} + \left(\Phi_3 + \Phi_4 \left(A_1 + A_2\right)\right) B\epsilon_t. \tag{14}$$

From here, this lemma follows:

Lemma 1 *A transformation of the vector of endogenous variables W_t given by $Z_t \equiv (W_t - \Theta W_{t-1})$ to exclude the backward-looking terms in (10) can be attained if a square matrix Θ exists such that:*

$$P(\Theta) = \Phi_2\Theta^2 - \Theta + \Phi_1 = \mathbf{0}_m, \tag{15}$$

where $\mathbf{0}_m$ is an $m \times m$ matrix of zeroes.

Binder and Pesaran (1995) and Binder and Pesaran (1997) establish the necessary and sufficient conditions under which a real-valued solution for Θ exists satisfying the companion quadratic matrix equation in (15) shown in Lemma 1. Given that, the vector of transformed endogenous variables, Z_t, follows a first-order forward-looking expectational difference system of this form:

$$\Gamma_0 Z_t = \Gamma_1 \mathbb{E}_t \left[Z_{t+1}\right] + \Gamma_2 X_{t|t-1} + \Gamma_3 \epsilon_t, \tag{16}$$

where $\Gamma_0 \equiv (\mathbf{I}_m - \Phi_2\Theta)$ and $\Gamma_1 \equiv \Phi_2$ are conforming $m \times m$ square matrices while $\Gamma_2 \equiv (\Phi_3 + \Phi_4 A_1)$ and $\Gamma_3 \equiv (\Phi_3 + \Phi_4 (A_1 + A_2)) B$ are the corresponding $m \times k$ matrices.

Proposition 2 in Binder and Pesaran (1997) provides sufficient conditions under which the matrix Γ_0 (which depends on the solution Θ to (15)) is nonsingular and invertible. Whenever Γ_0 is indeed nonsingular, the forward-looking expectational difference system in (16) can be re-stated as:

$$Z_t = F \mathbb{E}_t \left[Z_{t+1}\right] + G X_{t|t-1} + H \epsilon_t, \tag{17}$$

where $F \equiv (\Gamma_0)^{-1} \Gamma_1 = (\mathbf{I}_m - \Phi_2\Theta)^{-1} \Phi_2$ is an $m \times m$ square matrix while $G \equiv (\Gamma_0)^{-1} \Gamma_2 = (\mathbf{I}_m - \Phi_2\Theta)^{-1} (\Phi_3 + \Phi_4 A_1)$ and $H \equiv (\Gamma_0)^{-1} \Gamma_3 = (\mathbf{I}_m - \Phi_2\Theta)^{-1} (\Phi_3 + \Phi_4 (A_1 + A_2)) B$ are $m \times k$ matrices. Notice here that $G = H$ if $A_2 = \mathbf{0}_k$, but more generally it should follow that $H = (G + H_1 A_2) B$ where $H_1 \equiv (\Gamma_0)^{-1} \Phi_4 = (\mathbf{I}_m - \Phi_2\Theta)^{-1} \Phi_4$.

The Purely Forward-Looking Solution. If a solution to the forward-looking part of the LRE model given by (11) and (17) exists and is unique, then the reduced-form solution of the vector of transformed endogenous variables Z_t and the forcing variables vector re-expressed in terms of $X_{t+1|t}$ can be written in linear state-space form as:

$$X_{t+1|t} = A_1 X_{t|t-1} + (A_1 + A_2) B\epsilon_t, \tag{18}$$

$$Z_t = C X_{t|t-1} + D\epsilon_t, \tag{19}$$

where C and D are the same real-valued $m \times k$ matrices that describe the reduced-form solution for the vector of untransformed endogenous variables W_t in (12) and A_1, A_2, and B are the same $k \times k$ matrices that describe the stochastic process for the vector of forcing variables in (3).

Here, equations (18) and (19) are the corresponding ABCD representation of the solution of a purely forward-looking LRE model studied by Fernández-Villaverde et al. (2007) and Morris (2016). Equation (19) simply indicates that the reduced-form solution for the vector of transformed endogenous variables Z_t can be expressed as a linear mapping of the lagged exogenous forcing variables given by $X_{t|t-1}$ and the vector of shock innovations ϵ_t. Equation (18) simply recalls the dynamics of the redefined vector of forcing variables $X_{t+1|t}$ in (11).

Assuming the $m \times k$ matrix C has full column rank ($rank\,(C) = k$), a $k \times m$ left inverse matrix C_L^{-1} exists such that $C_L^{-1}C = \mathbf{I}_k$. If $m = k$, then C is invertible and its unique inverse matrix C^{-1} satisfies that $C^{-1}C = CC^{-1} = \mathbf{I}_k$. Accordingly, the left-inverse is also unique and must follow that $C_L^{-1} = C^{-1}$. If $m > k$, the left-inverse exists but is not unique (see, e.g., Theorem 2.1.1 of Rao and Mitra (1971)). Given that C is assumed to have linearly independent columns, the Moore-Penrose pseudo-inverse C^+ can be computed as $C^+ = (C^*C)^{-1}\,C^*$ where C^* denotes the Hermitian (or conjugate) transpose and C^*C is invertible. This pseudoinverse matrix C^+ is a useful way to obtain a left inverse matrix for C given that it naturally satisfies that $C^+C = \mathbf{I}_k$.

If a left inverse of C exists, equation (18) can be re-written as:

$$X_{t+1|t} = A_1 C_L^{-1} C X_{t|t-1} + (A_1 + A_2)\, B\epsilon_t. \tag{20}$$

Then, replacing $CX_{t|t-1} = Z_t - D\epsilon_t$ from (19) into (20), it follows that:

$$X_{t+1|t} = A_1 C_L^{-1}\,(Z_t - D\epsilon_t) + (A_1 + A_2)\, B\epsilon_t, \tag{21}$$

and, pre-multiplying (21) with C, that:

$$Z_{t+1} - D\epsilon_{t+1} = CX_{t+1|t} = CA_1 C_L^{-1} Z_t + \left(C\,(A_1 + A_2)\, B - CA_1 C_L^{-1} D\right)\epsilon_t. \tag{22}$$

As a result, a straightforward re-arranging of (22) gives a VARMA$(1,1)$ form for the vector of transformed endogenous variables Z_t. That is, it follows that:

$$Z_t = CA_1 C_L^{-1} Z_{t-1} + D\epsilon_t + \left(C\,(A_1 + A_2)\, B - CA_1 C_L^{-1} D\right)\epsilon_{t-1}. \tag{23}$$

Similar to Morris (2016), equation (23) allows casting the linear state-space form of the purely forward-looking LRE model solution into a VARMA$(1,1)$ representation.

Up to this point, the existence of a VARMA representation derived from the linear state-space in (20) – (19) requires only the left-invertibility of the matrix C while uniqueness necessitates additionally that $m = k$. This is so, irrespective of the relation between the matrices C and D in the linear state-space representation and the structural matrices F, G, H, A_1, A_2, and B imposed by theory through the purely forward-looking part of the expectational difference system in (17) and (11). Now, using (19) shifted one period ahead to replace Z_{t+1} in the forward-looking system given in (17), I obtain that:

$$Z_t = FCX_{t+1|t} + GX_{t|t-1} + H\epsilon_t. \tag{24}$$

In other words, this shows that the vector of transformed endogenous variables Z_t that solves the forward-looking part of the LRE model is a linear mapping of the vectors describing the redefined forcing variables, $X_{t+1|t}$ and $X_{t|t-1}$, and the vector of shock innovations, ϵ_t.

From (24) it also follows that:

$$Z_t = \left[FC + GA_1^{-1}\right] A_1 X_{t|t-1} + \left[FC\left(A_1 + A_2\right) B + H\right] \epsilon_t, \tag{25}$$

which arises after replacing $X_{t+1|t}$ out with (18). By the method of undetermined coefficients of Christiano (2002), matching the coefficients between (19) and (25) means that the unknown matrices that characterize the linear state-space representation of the forward-looking part of the LRE model solution must satisfy the following cross-equation restrictions:

$$C = \left[FC + GA_1^{-1}\right] A_1 = FCA_1 + G, \tag{26}$$

and

$$D = \left[FC\left(A_1 + A_2\right) B + H\right] = CA_1^{-1}\left(A_1 + A_2\right) B - TB, \tag{27}$$

where I define the matrix T as $T \equiv \left(\Gamma_0\right)^{-1} \Phi_3 A_1^{-1} A_2$. These equations relate C and D to the structural matrices F, G, and T as well as to the matrices A_1, A_2, and B that describe the stochastic process for the forcing variables.

Moreover, imposing the cross-equation restrictions in (27) also on the MA part of the VARMA$(1,1)$ in (23) implies that:

$$\left(C\left(A_1 + A_2\right) B - CA_1 C_L^{-1} D\right) = \mathbf{0}_{m\times k} + CA_1 C_L^{-1} TB, \tag{28}$$

and from here it follows that:

$$Z_t = CA_1 C_L^{-1} Z_{t-1} + \left(CA_1^{-1}\left(A_1 + A_2\right) - T\right) B\epsilon_t + CA_1 C_L^{-1} TB\epsilon_{t-1}. \tag{29}$$

Replacing equation (18) re-written as $X_{t|t-1} = A_1^{-1}\left(X_{t+1|t} - \left(A_1 + A_2\right) B\epsilon_t\right)$ into (19), it holds that:

$$Z_t = CA_1^{-1} X_{t+1|t} + \left[D - CA_1^{-1}\left(A_1 + A_2\right) B\right] \epsilon_t = CA_1^{-1} X_{t+1|t} - TB\epsilon_t, \tag{30}$$

which follows from the cross-equation restrictions that characterize the matrix D in (27). Hence, the linear mapping of the vector of exogenous forcing variables $X_{t+1|t}$ into transformed endogenous variables Z_t in (24) can be re-written and simplified as in (30), but still depends on the vector of shock innovations.

This comes to show that when the forcing variables are driven by a VAR(1) stochastic process (i.e., $A_2 = \mathbf{0}_k$), then $T = \mathbf{0}_{m\times k}$ and accordingly the forward-looking part of the solution in (29) naturally inherits its VAR(1) form from that of the stochastic process for the forcing variables. In general, when the forcing variables behave as a VARMA$(1,1)$ stochastic process, the forward-looking part of the solution also inherits a VARMA$(1,1)$ representation. However, whenever the structural relationships of the LRE model in (1) can be cast in such a way that only $X_{t+1|t}$ enters into the specification (i.e., when $\Phi_3 = \mathbf{0}_{m\times k}$), then $T = \mathbf{0}_{m\times k}$ and once again the forward-looking part of the solution can be described with a VAR(1) form even when the forcing variables follow a VARMA$(1,1)$.

Accordingly, re-writing the cross-equation restrictions that pin down C in (26), the characterization of the forward-looking part of the LRE solution in (29) can be obtained with the following lemma:

Lemma 2 *Assumption 1 and Assumption 2 hold and Γ_0 is invertible. The matrix C solves the companion Sylvester matrix equation:*

$$FC + C\left(-A_1^{-1}\right) = -GA_1^{-1}, \tag{31}$$

and the matrix D is the linear transformation of C given by (27) where F, G, H, T, A_1, A_2, and B are the matrices that describe the structural relationships of the purely forward-looking part of the LRE model and the stochastic processes in (3) and (17).

(a) If a matrix C that solves (31) exists, the matrix D pinned down by (27) also exists and they both characterize a linear state-space of the form given in (18) − (19).

(b) If a matrix C that solves (31) is also left invertible, a solution for the vector of transformed endogenous variables Z_t that follows the VARMA$(1,1)$ representation given by (29) can be obtained using the pseudoinverse (that is, using $C_L^{-1} = C^{+} = \left(C^{*}C\right)^{-1} C^{*}$ where C^{*} is the Hermitian transpose).

(c) If it also holds that $m = k$, then the VARMA$(1,1)$ representation given by (29) not only exists but it is also unique (and $C_L^{-1} = C^{-1}$). Moreover, if the stochastic process in (3) is a VAR(1) process (i.e., $A_2 = \mathbf{0}_k$) or the structural relations in (17) depend on $X_{t+1|t}$ and not on X_t (i.e., $\Phi_3 = \mathbf{0}_{m\times k}$), then the unique solution given by (29) takes the following VAR(1) form:

$$Z_t = CA_1C_L^{-1}Z_{t-1} + CA_1^{-1}\left(A_1 + A_2\right)B\epsilon_t. \tag{32}$$

The proof of this lemma follows directly from the implications of the cross-equation restrictions in (26) − (27), as discussed above. It should be noted here that Corollary 3 of Morris (2016) similarly shows that the linear state-space model in (18) − (19) can be cast as a VAR(1). The logic of Morris (2016) is similar to the one I follow to derive equation (32). The contribution of this paper is to exploit the cross-equation restrictions of the LRE model to characterize the mapping in terms of the structural matrices of the LRE model itself. Hence, Lemma 2 provides a simple way to both determine the existence and uniqueness of the VARMA$(1,1)$/VAR(1) representation given by (29)/(32) and a simple way to characterize that solution from the matrices F, G, H, T, A_1, A_2, and B. Moreover, the paper also shows next that, by decoupling the forward- and backward-looking parts of the LRE model solution, the results of Lemma 2 can then be easily extended to richer models whose solution can be cast in the linear state-space form given by (11) and (12).

2.3 The Canonical First-Order LRE Model Solution

The solution of the full canonical (first-order) system of equations from the LRE model in (1) and (3) can be recovered combining its backward-looking part, ΘW_{t-1}, and its forward-looking part, $Z_t \equiv (W_t - \Theta W_{t-1})$. Simply replacing $Z_t \equiv (W_t - \Theta W_{t-1})$ into (29), it follows that:

$$(W_t - \Theta W_{t-1}) = CA_1C_L^{-1}\left(W_{t-1} - \Theta W_{t-2}\right) + \left(CA_1^{-1}\left(A_1 + A_2\right) - T\right)B\epsilon_t + CA_1C_L^{-1}TB\epsilon_{t-1}. \tag{33}$$

Hence, the dynamics of the vector of endogenous variables W_t can be described with this lemma:

Lemma 3 *If Assumption 1, Assumption 2, and the conditions associated with Lemma 1 and Lemma 2 hold, it follows that:*

(a) The VARMA$(2,1)$ representation of the first-order canonical LRE model solution for the vector of

endogenous variables W_t, when one exists, is given by:

$$W_t = \Psi_1 W_{t-1} + \Psi_2 W_{t-2} + \Psi_3 \epsilon_t + \Psi_4 \epsilon_{t-1},\ \epsilon_t \sim i.i.d.\left(0, \mathbf{I}_k\right), \tag{34}$$

where $\Psi_1 \equiv \left(\Theta + CA_1 C_L^{-1}\right)$ and $\Psi_2 \equiv -CA_1 C_L^{-1}\Theta$ are $m \times m$ square matrices while $\Psi_3 \equiv \left(CA_1^{-1}\left(A_1 + A_2\right) - T\right) B$ and $\Psi_4 \equiv CA_1 C_L^{-1} TB$ are $m \times k$ matrices. If it also holds that $m = k$, the VARMA$(2,1)$ representation in (34) is also unique.

(b) Whenever $A_2 = \mathbf{0}_k$ or $\Phi_3 = \mathbf{0}_{m \times k}$, the VARMA$(2,1)$ representation in (34) reduces to a VAR(2) of the following form:

$$W_t = \Psi_1 W_{t-1} + \Psi_2 W_{t-2} + e_t,\ e_t \sim i.i.d.\left(0, \Omega\right),\ \Omega \equiv \Psi_3^T \Psi_3, \tag{35}$$

where $\Psi_1 \equiv \left(\Theta + CA_1 C_L^{-1}\right)$ and $\Psi_2 \equiv -CA_1 C_L^{-1}\Theta$ are the same $m \times m$ square matrices but $\Psi_4 = \mathbf{0}_{m \times k}$ and the $m \times k$ matrix Ψ_3 reduces to $\Psi_3 \equiv CA_1^{-1}\left(A_1 + A_2\right) B$. The VAR$(2)$ residual innovations $e_t \equiv \Psi_3 \epsilon_t$ in (35) are a rotation of the shock innovations ϵ_t. If it also holds that $m = k$, then the the VAR(2) representation of the LRE model solution in (35) not only exists but is also unique.

(c) The 'poor man's invertibility condition' of Fernández-Villaverde et al. (2007) can also be trivially verified and the VAR(2) representation in (35) satisfies Hansen and Sargent (1980)'s fundamentalness property. Whenever (35) holds with $A_2 \neq \mathbf{0}_k$, this would also require that $rank\left(A_1 + A_2\right) = k$.

The proof of the VARMA representation in (34) and the finite-order VAR form in (35) as well as its uniqueness follows directly from an algebraic manipulation of (33) and Lemma 2.

The 'poor man's invertibility condition' assumes that the number of endogenous variables in W_t is equal to the number of exogenous forcing variables in X_t (that is, $m = k$) and holds whenever all eigenvalues of $\left(A_1 - \left(A_1 + A_2\right) BD^{-1} C\right)$ ($k \times k$ matrix) lie inside the unit circle. As indicated by part (b) of Lemma 2, whenever the matrix T satisfies that $T = \mathbf{0}_{k \times k}$, D is invertible if both the solution C to the companion Sylvester matrix equation in (31) and $\left(A_1 + A_2\right)$ have full column rank (that is, when C and $\left(A_1 + A_2\right)$ are invertible).[5] Assuming that all of this indeed holds and replacing D out with the cross-equation restrictions in (27), it follows that $\left(A_1 - \left(A_1 + A_2\right) BD^{-1} C\right) = \left(A_1 - \left(A_1 + A_2\right) BB^{-1}\left(A_1 + A_2\right)^{-1} A_1 C_1^{-1} C\right) = \mathbf{0}_k$. Hence, the 'poor man's invertibility condition' of Fernández-Villaverde et al. (2007) is trivially satisfied, i.e., has all its eigenvalues inside the unit circle.

The LRE model solution in finite-order VAR form given by (35) can be re-written to express the vector of shock innovations ϵ_t in terms of the observable vector of endogenous variables W_t as follows:

$$\epsilon_t = \Psi_3^{-1}\left[W_t - \Psi_1 W_{t-1} - \Psi_2 W_{t-2}\right], \tag{36}$$

where Ψ_3 is invertible if indeed C and $\left(A_1 + A_2\right)$ are invertible. Hence, the shock innovations can be recovered from the finite-order VAR form for all $t \geq 0$ given initial conditions W_{-1} and W_{-0}, that is, the fundamentalness property holds in the sense of Hansen and Sargent (1980).

[5] Notice that $T = \mathbf{0}_k$ if the stochastic process for the vector of forcing variables is a VAR(1) (that is, $A_2 = \mathbf{0}_k$). In this case, the inverse $\left(A_1 + A_2\right)^{-1} = A_1^{-1}$ is well-defined given Assumption 1. If instead $T = \mathbf{0}_k$ because $\Phi_3 = \mathbf{0}_{m \times k}$, then Assumption 1 is not sufficient to ensure that the inverse $\left(A_1 + A_2\right)^{-1}$ exists because the sum of two invertible matrices is not necessarily invertible since $rank\left(A_1 + A_2\right) \leq k$ holds but not necessarily always with equality. A sufficient condition would be that A_1 and A_2 be also positive definite because, if so, the sum is also positive definite and invertible.

Some additional conditions regarding the properties of the matrix Θ that solves the companion quadratic matrix equation in (15) in Lemma 1 ensure the companion finite-order VAR form of the LRE model solution in (35) is well-behaved:

Corollary 1 *The unique VAR(2) representation for the vector of endogenous variables W_t in (35) characterized by Lemma 3 where $m = k$ satisfies the following properties:*

(a) If the matrix Θ that solves (15) in Lemma 1 has all its eigenvalues inside the unit circle (that is, Θ is invertible), the VAR(2) process can be shown to be stationary.

(b) Under the prevailing assumptions in Lemma 3, the variance-covariance matrix Ω is symmetric and positive definite.

The VAR(2) representation in (35) can be re-cast in companion VAR(1) form as follows:

$$\begin{pmatrix} W_t \\ W_{t-1} \end{pmatrix} = \begin{pmatrix} \Psi_1 & \Psi_2 \\ \mathbf{I}_k & \mathbf{0}_k \end{pmatrix} \begin{pmatrix} W_{t-1} \\ W_{t-2} \end{pmatrix} + \begin{pmatrix} e_t \\ 0_{k\times 1} \end{pmatrix}, \tag{37}$$

where $\mathbf{I}_k$ is the $k \times k$ identity matrix, $\mathbf{0}_k$ is a $k \times k$ matrix of zeroes as noted before, and $\mathbf{0}_{k\times 1}$ is a $k \times 1$ column-vector of zeroes. The VAR(2) in (35) is stationary if the VAR(1) specification in (37) is shown to be stationary, i.e., if all eigenvalues of $\overline{\Psi} \equiv \begin{pmatrix} \Psi_1 & \Psi_2 \\ \mathbf{I}_k & \mathbf{0}_k \end{pmatrix}$ are inside the unit circle. Using the Schur complement method, the determinant of the block matrix $\overline{\Psi} - \lambda \mathbf{I}_{2k}$ (where $\mathbf{I}_{2k}$ is the $2k \times 2k$ identity matrix) can be expressed as $\det\left(\overline{\Psi} - \lambda \mathbf{I}_{2k}\right) = \det\left(\lambda^2 \mathbf{I}_k - \lambda \Psi_1 - \Psi_2\right)$. From here, using the definitions of Ψ_1 and Ψ_2 given in Lemma 3, it follows that:

$$\begin{aligned} \det\left(\overline{\Psi} - \lambda \mathbf{I}_{2k}\right) &= \det\left(\lambda^2 \mathbf{I}_k - \lambda\left(\Theta + CAC^{-1}\right) + CAC^{-1}\Theta\right) \\ &= \det\left(\Theta - \lambda \mathbf{I}_k\right) \det\left(CAC^{-1} - \lambda \mathbf{I}_k\right). \end{aligned} \tag{38}$$

Thus, this suffices to show that the eigenvalues of the block matrix $\overline{\Psi}$ are the combined eigenvalues of Θ and CAC^{-1}.

The matrix CA_1C^{-1} is similar to the matrix A_1 so long as the solution C to the companion Sylvester matrix equation in (31) exists and is invertible, as implied by Lemma 2. Two similar matrices have the same eigenvalues and, therefore, Assumption 1 which establishes that the eigenvalues of A_1 are all inside the unit circle implies that all eigenvalues of CA_1C^{-1} are inside the unit circle as well. Therefore, as indicated in Corollary 1, the stationarity of the stochastic process in (35) requires solely that the eigenvalues of the solution Θ to the companion quadratic matrix equation in (15) in Lemma 1 also lie inside the unit circle.

The matrix $\Psi_3 \equiv CA_1^{-1}\left(A_1 + A_2\right)B$ is $k \times k$ square. The variance-covariance matrix is $\Omega \equiv \Psi_3^T \Psi_3$ and its transpose is $\Omega^T = \left(\Psi_3^T \Psi_3\right)^T$. The transpose of a product is the product of the transposes in the opposite order, therefore it follows that $\Omega^T = \Psi_3^T \left(\Psi_3^T\right)^T$. Furthermore, by the properties of the transpose it is true that $\left(\Psi_3^T\right)^T = \Psi_3$ and, accordingly, it should hold that $\Omega^T = \Psi_3^T \left(\Psi_3^T\right)^T = \Omega$. Thus, the variance-covariance matrix Ω is indeed symmetric.

Under Assumption 1 and Assumption 2, $rank\left(A_1\right) = rank\left(B\right) = k$ and by implication also holds true that $rank\left(A^{-1}\right) = rank\left(B^{-1}\right) = k$. As noted before in regards to Lemma 2, a unique solution in finite-order VAR form exists if the matrix C that solves the companion Sylvester equation in (31) is invertible and has

$rank\left(C\right) = k$. Given this, it is straightforward to show that $rank\left(\Psi_3\right) = k$ and accordingly the k columns of $\Psi_3 \equiv CA^{-1}\left(A_1 + A_2\right)B$ are linearly independent when $rank\left(A_1 + A_2\right) = k$.

Using the Gram-Schmidt orthonormalization method, there is a $k \times k$ invertible matrix Q such that the columns of $\Psi_3 Q$ are a family of k orthonormal vectors such that $\left(\Psi_3 Q\right)^T\left(\Psi_3 Q\right) = \mathbf{I}_k$. Let $x \in \mathbb{R}^k \setminus \{0\}$ and, from $Q^{-1}x \neq 0$, it follows that $\left\|Q^{-1}x\right\| > 0$. From here, $x^T\left(\Psi_3^T\Psi_3\right)x = x^T\left(\Psi_3 QQ^{-1}\right)^T\left(\Psi_3 QQ^{-1}\right)x = \left(Q^{-1}x\right)^T\left(\left(\Psi_3 Q\right)^T\left(\Psi_3 Q\right)\right)\left(Q^{-1}x\right) = \left(Q^{-1}x\right)^T\mathbf{I}_k\left(Q^{-1}x\right) = \left\|Q^{-1}x\right\| > 0$. Being x arbitrary, it naturally must hold that:

$$\forall x \in \mathbb{R}^k \setminus \{0\},\ x^T\left(\Psi_3^T\Psi_3\right)x > 0, \tag{39}$$

and, therefore, the variance-covariance matrix Ω is shown to be positive definite (and that, in turn, ensures it is invertible as well).

In short, Corollary 1 implies that the finite-order VAR representation in (35) that characterizes the unique solution—when one exists—to the first-order canonical LRE model given by (1) and (3) is a well-behaved stochastic process that can be related to the structure of the model given by the matrices F, G, H, and T (tied to Φ_1, Φ_2, Φ_3, Φ_4) together with A_1, A_2, and B, under some conditions. Taking as given Assumption 1 and Assumption 2, the required conditions are that $m = k$ as well as the invertibility of the matrix C that solves the companion Sylvester matrix equation in (31) and the invertibility of the matrix Θ that solves the companion quadratic matrix equation in (15) if $A_2 = \mathbf{0}_k$ plus the invertibility of $\left(A_1 + A_2\right)$ if $\Phi_3 = \mathbf{0}_{m\times k}$.[6]

The key to the results of this paper is two-fold. First, I take advantage of the cross-equation restrictions that link the matrices of the linear state-space representation to the structural matrices of the LRE model. That allows me to recover a VAR form, but also provides a way to characterize it. Second, using the canonical representation of the VARMA process for the vector of forcing variables and decoupling the forward-looking part from the backward-looking part of the LRE model, I am able to investigate a more general structure that includes lags of the endogenous variables and moving average terms on the process for the forcing variables. Indeed, this idea of simplifying richer models to make them amenable to be cast in finite-order VAR form can be pushed even further noting that the canonical first-order LRE model in (1) and (3) can be generalized with more than one lead and one lag of the endogenous variables in W_t and of the forcing variables in X_t, as shown elsewhere in Broze et al. (1985) and Broze et al. (1990).

2.4 Local Identification of the Finite-Order VAR

The elements of the matrices Φ_1, Φ_2, Φ_3, Φ_4, A_1, A_2, and B that describe the structural relationships and the dynamics of the forcing variables for the canonical (first-order) LRE model in (1) and (3) are each a function of a vector of structural parameters δ, where δ contains $j > 0$ elements and lies in the space of admissible parameter values $\Delta \subset \mathbb{R}^j$. Lemma 3 and Corollary 1 establish the mapping between the matrices Φ_1, Φ_2, Φ_3, Φ_4, A_1, A_2, and B that characterize the LRE model and the matrices Ψ_1, Ψ_2, and Ω that describe its reduced-form solution in finite-order VAR form given by (35). For a given vector δ, it follows that if the finite-order VAR representation in (35) exists and is unique (where $m = k$), then Ψ_1, Ψ_2, and Ω completely characterize the dynamics of the endogenous variables in W_t. I define $\tau \equiv \left[vec\left(\Psi_1\right)^T, vec\left(\Psi_2\right)^T, vech\left(\Omega\right)\right]^T$ as the vector collecting all the elements of Ψ_1, Ψ_2, and Ω, where $vec\left(\cdot\right)$ refers to the vector of columns of a

[6] This finite-order VAR representation in (35) is obtained for the linear state-space solution given by (12) and (3). It is worth noting that the VAR form is only considered by Morris (2016) under the conditions of his Corollary 3 which amount here to the special case where $\Theta = \mathbf{0}_k$.

given matrix and $vech(\cdot)$ refers to the same operation except that for each column only that part which is on or below the diagonal of the given matrix is kept.[7] Hence, the vector τ contains $2k^2 + k\left(\frac{k+1}{2}\right)$ distinct elements.

It follows from (35) that the unconditional second moments of W_t are given by the $k \times k$ matrix of autocovariances $\Sigma(s) \equiv \mathbb{E}\left(W_t W_{t+s}^T\right)$ for any s such that:

$$\Sigma(s) = \begin{cases} \Psi_1 \Sigma(0) \Psi_1^T + \Psi_1 \Sigma(1) \Psi_2^T + \Psi_2 \Sigma(1) \Psi_1^T + \Psi_2 \Sigma(0) \Psi_2^T + \Omega, & \text{if } s = 0, \\ \Psi_2 \Sigma(s-1) + \Psi_1 \Sigma(s-2), & \text{for any other } s \neq 0, \end{cases} \tag{40}$$

which satisfies that $\Sigma(s) \equiv \mathbb{E}\left(W_t W_{t+s}^T\right) = \mathbb{E}\left(W_t W_{t-s}^T\right) \equiv \Sigma(-s)$. If I denote the sample size of the observed endogenous variables W_t as T, then the vector of unique elements that describes the variance-covariance matrix for the observed data $\left[W_1^T, W_2^T, ..., W_T^T\right]^T$ can be defined as $\sigma_T \equiv \left[vech\left(\Sigma(0)^T\right), vech\left(\Sigma(1)^T\right), ..., vech\left(\Sigma(T-1)^T\right)\right]^T$. This is a vector of $k\left(\frac{k+1}{2}\right) + (T-1)k^2$ elements collecting all the distinct coefficients that determine the second moments of the observed data sample. Iskrev (2010) further generalizes this to include, for instance, the joint estimation of the steady state using also the first moments of the data. If e_t is Gaussian, σ_T together with the means of the data contains all the model-implied information that can be used for estimation purposes. Assuming that τ is unique for each admissible value of δ, then σ_T is a function of δ as well. Accordingly, Iskrev (2010) shows that the j parameters of δ are locally identifiable at δ_0 from the sample likelihood in a limited information setting if the Jacobian $J(\delta_0) = \frac{\partial \sigma_T}{\partial \delta^T}\Big|_{\delta=\delta_0}$ has full column rank (that is, if $rank(J(\delta_0)) = j$). On this point see also Komunjer and Ng (2011), Qu and Tkachenko (2012), and also the in-depth discussions in Morris (2016) and Morris (2017).

3 An Illustration with a Monetary Model

In this section, I illustrate the finite-order VAR representation articulated in (35) in the context of a simplified version of the medium-scale New Keynesian model of Smets and Wouters (2003) and Smets and Wouters (2007), as described in Martínez-García (2018). This application is similar to that studied in Morris (2016) or that, more extensively analyzed, by Morris (2017). It illustrates the practical significance of mapping LRE solutions into exact finite-order VARs not just for theoretical, but also for applied empirical research too.

3.1 The Workhorse New Keynesian Model

The model includes a dynamic Investment-Saving (IS) equation with external habit formation given by:

$$y_t = \frac{1}{1+h}\mathbb{E}_t(y_{t+1}) + \frac{h}{1+h}y_{t-1} - \frac{1-h}{(1+h)\sigma_c}\left(i_t - \mathbb{E}_t(\pi_{t+1}) - (1-\rho_b)\varepsilon_t^b\right), \tag{41}$$

where $0 \leq h < 1$ is the habit persistence parameter and σ_c determines the inverse of the intertemporal elasticity of substitution. The dynamics of the preference shock ε_t^b given by:

$$\varepsilon_t^b = \rho_b \varepsilon_{t-1}^b + \sigma_\zeta \zeta_t, \tag{42}$$

[7] Hence, using $vech(\Omega)$ means that I retain only the distinct elements of the variance-covariance matrix Ω.

act as a shifter of the IS equation. The innovation ζ_t is assumed to be i.i.d.$(0,1)$ white noise with $-1 < \rho_b < 1$ and $\sigma_\zeta > 0$ being the corresponding persistence and volatility parameters, respectively.

The hybrid Phillips curve equation relates inflation π_t to output y_t as follows:

$$\pi_t = \gamma_f \mathbb{E}_t\left(\pi_{t+1}\right) + \gamma_b \pi_{t-1} + \kappa\left[\left(\sigma_l + \sigma_c \frac{1}{1-h}\right) y_t - \sigma_c \frac{h}{1-h} y_{t-1} + \varepsilon_t^l - \left(1+\sigma_l\right)\varepsilon_t^a\right], \tag{43}$$

where $\gamma_f \equiv \frac{\beta}{1+\beta\gamma_p} > 0$ and $\gamma_b \equiv \frac{\gamma_p}{1+\beta\gamma_p} \geq 0$ satisfy $\gamma_f + \gamma_b \leq 1$ and the composite coefficient $\kappa \equiv \frac{1}{1+\beta\gamma_p}\frac{\left(1-\beta\xi_p\right)\left(1-\xi_p\right)}{\xi_p} \geq 0$ defines the slope of the hybrid Phillips curve. Here, $0 < \beta < 1$ is the intertemporal discount factor, $\sigma_l > 0$ is the inverse of the Frisch elasticity of labor, ξ_p is the Calvo (1983) parameter that sets the degree of price stickiness, and the parameter $0 \leq \gamma_p \leq 1$ regulates the pass-through from previous period inflation.

The exogenous productivity shock process ε_t^a and the labor supply shock ε_t^l behave as follows:

$$\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \sigma_\delta \delta_t, \tag{44}$$

$$\varepsilon_t^l = \rho_l \varepsilon_{t-1}^l + \sigma_\nu \nu_t, \tag{45}$$

and enter as shifters (also referred as cost-push shocks) of the hybrid Phillips curve. The innovations δ_t and ν_t are i.i.d.$(0,1)$ white noise. The corresponding persistence parameters are $-1 < \rho_a, \rho_l < 1$, while the volatility parameters are $\sigma_\delta, \sigma_\nu > 0$.

The model is completed with a standard Taylor (1993) rule with inertia as follows:

$$i_t = \rho_i i_{t-1} + \left(1-\rho_i\right)\left[\psi_\pi \pi_t + \psi_y\left(y_t - y_t^n\right)\right] + \varepsilon_t^m, \tag{46}$$

where the policy rate i_t responds to the output gap $(y_t - y_t^n)$ as well as to inflation π_t. The corresponding policy parameters satisfy that $\psi_\pi > 1$ and $\psi_y \geq 0$ while the policy inertia is modeled with the parameter $0 \leq \rho_i < 1$. The associated monetary policy shock ε_t^m follows an exogenous process of the following form:

$$\varepsilon_t^m = \rho_m \varepsilon_{t-1}^m + \sigma_\xi \xi_t, \tag{47}$$

where ξ_t is i.i.d.$(0,1)$ white noise, the persistence parameter is $-1 < \rho_m < 1$ and the volatility parameter is given by $\sigma_\xi > 0$.

Finally, the potential output y_t^n is the level of economic activity that would prevail absent all frictions (nominal rigidities). A standard log-linearization of the labor supply and labor demand equations around the steady state under flexible prices yields:

$$y_t^n = \frac{\sigma_c h}{\sigma_l\left(1-h\right)+\sigma_c} y_{t-1}^n + \frac{\left(1-h\right)\left(1+\sigma_l\right)}{\sigma_l\left(1-h\right)+\sigma_c}\varepsilon_t^a - \frac{\left(1-h\right)}{\sigma_l\left(1-h\right)+\sigma_c}\varepsilon_t^l, \tag{48}$$

which is driven solely by the exogenous productivity shock ε_t^a and the labor supply shock ε_t^l.

Let me define the vector of endogenous variables as $W_t = \left(y_t, \pi_t, i_t, y_t^n\right)^T$, the forcing variables as $X_t = \left(\varepsilon_t^a, \varepsilon_t^b, \varepsilon_t^l, \varepsilon_t^m\right)^T$, and the vector of innovations as $\epsilon_t = \left(\delta_t, \zeta_t, \nu_t, \xi_t\right)^T$. In Table 1, I adopt a parameterization of the New Keynesian model largely based on the values estimated by Smets and Wouters (2003):

Table 1. Parameterization of the Workhorse New Keynesian Model

Structural Parameter	Notation	Value	Source
Discount factor	$0 < \beta < 1$	0.990	Smets and Wouters (2003)
Degree of price stickiness	$0 \leq \xi_p < 1$	0.905	Smets and Wouters (2003)
Degree of price indexation	$0 \leq \gamma_p \leq 1$	0.472	Smets and Wouters (2003)
▷ Forward-looking weight on Phillips curve	$\gamma_f \equiv \frac{\beta}{1+\beta\gamma_p}$	0.675	Composite
▷ Backward-looking weight on Phillips curve	$\gamma_b \equiv \frac{\gamma_p}{1+\beta\gamma_p}$	0.322	Composite
▷ Slope of the Phillips curve	$\kappa \equiv \frac{(1-\beta\xi_p)(1-\xi_p)}{(1+\beta\gamma_p)\xi_p}$	0.007	Composite
Inverse intertemporal elasticity of substitution	$\sigma_c > 0$	1.371	Smets and Wouters (2003)
Inverse of the Frisch elasticity of labor supply	$\sigma_l > 0$	2.491	Smets and Wouters (2003)
External habit formation parameter	$0 < h < 1$	0.595	Smets and Wouters (2003)
Monetary Policy			
Policy inertia	$0 \leq \rho_i < 1$	0.958	Smets and Wouters (2003)
Response to inflation deviations	$\psi_\pi > 1$	1.688	Smets and Wouters (2003)
Response to output gap deviations	$\psi_y \geq 0$	0.095	Smets and Wouters (2003)
Exogenous Shock Parameters			
Persistence of the productivity shock	$-1 < \rho_a < 1$	0.815	Smets and Wouters (2003)
Volatility of the productivity shock	$\sigma_\delta > 0$	0.345**	Smets and Wouters (2003)
Persistence of the preference shock	$-1 < \rho_b < 1$	0.842	Smets and Wouters (2003)
Volatility of the preference shock	$\sigma_\zeta > 0$	0.089**	Smets and Wouters (2003)
Persistence of the labor supply shock	$-1 < \rho_l < 1$	0.891	Smets and Wouters (2003)
Volatility of the labor supply shock	$\sigma_\nu > 0$	1.244**	Smets and Wouters (2003)
Persistence of the monetary shock	$-1 < \rho_m < 1$	0.750*	Parameterized
Volatility of the monetary shock	$\sigma_\xi > 0$	0.001**	Smets and Wouters (2003)

* Introduces persistence unlike the parameterization in Smets and Wouters (2003) ensuring also that the matrix A is invertible.

** The volatility parameters are chosen such that the standard deviation of each shock innovation recovered from the data is equal to 1.

3.2 Some Implications for Monetary VARs

Given the parameterization reported in Table 1, the Blanchard and Kahn (1980) conditions are satisfied and a solution exists and is unique. Moreover, a solution Θ to the companion quadratic matrix equation in (15) exists that has all its roots inside the unit circle and the solution C to the companion Sylvester matrix equation (31) exists and is invertible. Therefore, the unique solution of the workhorse New Keynesian model has a VAR(2) representation given by (35) where the corresponding composite coefficient matrices take the following form:

$$\Psi_1 = \begin{pmatrix} 1.5057 & -0.5355 & -3.2133 & -0.0607 \\ -0.0648 & 1.4821 & 0 & 0.0648 \\ -0.0039 & 0.0576 & 1.7171 & 0.0043 \\ 0.5847 & -1.9717 & -5.4962 & 0.7552 \end{pmatrix},$$
$$\Psi_2 = \begin{pmatrix} -0.5253 & 0.2655 & 2.4934 & 0.0179 \\ 0.0222 & -0.4768 & 0 & -0.0222 \\ 0.0006 & -0.0367 & -0.7254 & -0.0009 \\ -0.2740 & 1.0036 & 4.2124 & -0.1464 \end{pmatrix}, \tag{49}$$
$$\Psi_3 = \begin{pmatrix} 0.0282 & 0.0203 & -0.0405 & -0.0132 \\ -0.0263 & 0.0048 & 0.0312 & -0.0055 \\ -0.0026 & 0.0004 & 0.0029 & 0.0006 \\ 0.2050 & 0 & -0.2117 & 0 \end{pmatrix},$$

with non-zero entries almost everywhere. The variance-covariance matrix $\Omega \equiv \Psi_3^T \Psi_3$ is symmetric and positive definite and the VAR(2) specification is also stationary.

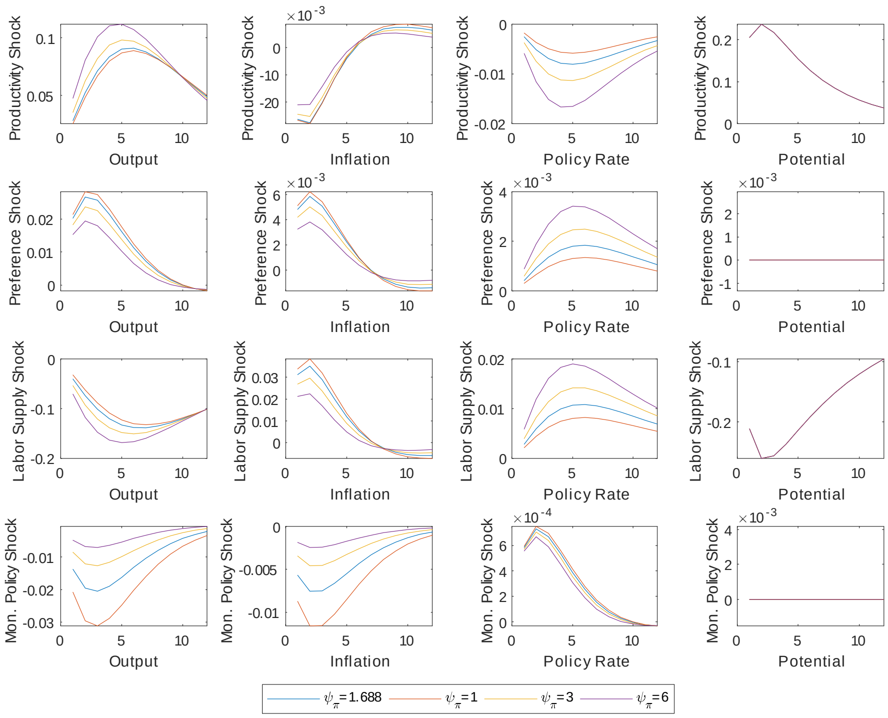
The matrices that characterize the companion finite-order VAR representation in (49) indicate already that any empirical evidence which hinges upon Cholesky (and, in general, zero) restrictions should be interpreted with caution as it imposes identification restrictions that may not have a structural interpretation tied to theory—to be more specific, a structural interpretation tied to the workhorse New Keynesian model—as noted, among others, by Carlstrom et al. (2009).

Figure 1 plots the theoretical one-standard deviation impulse response functions (IRFs) at different degrees of the anti-inflation bias ψ_π in order to showcase the potential changes in the transmission mechanism of monetary policy arising from the policymakers willingness to respond to inflation deviations. For starters, Figure 1 confirms that the paradoxical empirical finding that a monetary shock could lead to an increase in inflation (the "price puzzle") is inconsistent with the predictions of the workhorse New Keynesian model laid out here.

While the propagation of monetary shocks in this model appears inconsistent with the implications of a Cholesky (or zero-restriction) identification strategy, it is nonetheless consistent with the sign-restrictions proposed by Uhlig (2005). That is because the propagation of the monetary shock does not increase inflation nor lower the short-term interest rate for a number of periods after the shock. Furthermore, Figure 1 also shows that the decline on inflation and output resulting from a one-standard deviation positive (contractionary) shock to monetary policy gets attenuated the higher the anti-inflation bias stance on monetary policy ($\uparrow \psi_\pi$) becomes.

The findings in Figure 1 show that the transmission mechanism of structural shocks—not just monetary policy shocks—and their spillovers ultimately depend in nonlinear ways on the policy parameters of the Taylor (1993) rule. A higher anti-inflation bias stance on monetary policy ($\uparrow \psi_\pi$) tends to dampen the endogenous output response to preference and monetary policy shock innovations while it amplifies the response to productivity and labor supply shock innovations. In turn, the response of endogenous inflation to all types of shocks becomes more muted as the anti-inflation bias stance increases ($\uparrow \psi_\pi$). Inflation and output move in the same direction in response to preference shocks and monetary policy shocks, but they move in opposite directions in response to productivity shocks and labor supply shocks.[8]

Figure 1. Theoretical IRFs at Different Degrees of Anti-Inflationary Bias



Note: This figure displays the theoretical impulse response functions (IRFs) of the workhorse New Keynesian model keeping the parameterization invariant as in Table 1 except for the policy parameter ψ_π.

To sum up, the New Keynesian model laid out here suggests that popular identification strategies used in structural VAR studies impose restrictions on how monetary policy (or other structural) shocks affect macroeconomic variables that can be contrasted (and mapped) against a theory-implied VAR specification. The standard Cholesky identification, on the one hand, is shown to potentially distort the impulse response functions, and could be a contributing factor of a number of empirical anomalies such as the so-called "price puzzle" even when underneath it all the *true* data-generating process is the New Keynesian model.[9] On the other hand, sign-restrictions as those proposed by Uhlig (2005) for monetary policy shocks do better without having to impose all cross-equation restrictions implied by the model.

4 Concluding Remarks

I show that under some conditions the solution to a large class of LRE models—when one exists and is unique—can be represented in finite-order VAR form. An important contribution of the paper is to establish

[8] Additional analysis exploring the implications of this model can be found in Martínez-García (2018).

[9] The "price puzzle" refers to the paradoxical result that an empirically identified monetary shock leads to an increase in inflation which seems counterintuitive based on theory.

the conditions under which such a finite-order VAR representation of the solution exists, is well-behaved, and satisfies the fundamentalness property of Hansen and Sargent (1980). This, in turn, is tied to the solution of a well-known companion quadratic matrix equation and a companion Sylvester matrix equation. The paper is complemented with a number of Matlab routines and codes for the solution of those companion matrix equations and the characterization of the finite-order VAR representation of the LRE model solution.

The economic-significance of the finite-order VAR mapping arising from theory is illustrated studying the transmission mechanism of monetary policy with a variant of the workhorse New Keynesian model. For this purpose, the paper analyzes the finite-order VAR representation of the model solution with particular attention paid to the role of the anti-inflation stance of the monetary policy rule. The paper shows how the identification of fundamental shocks for empirical research—including the recovery of monetary shocks—can be made tractable using the cross-equation restrictions obtained from the model. In contrast, it also shows that standard identification assumptions (Cholesky, zero-restrictions, etc.) used in the estimation of related structural VARs present significant shortcomings.

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