Federal Reserve Bank of Dallas

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Working Paper 2302 **April 2023**

Research Department

https://doi.org/10.24149/wp2302

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April 11, 2023

Abstract

Do state-dependent local projections asymptotically recover the population responses of macroeconomic aggregates to structural shocks? The answer to this question depends on how the state of the economy is determined and on the magnitude of the shocks. When the state is exogenous, the local projection estimator recovers the population response regardless of the shock size. When the state depends on macroeconomic shocks, as is common in empirical work, local projections only recover the conditional response to an infinitesimal shock, but not the responses to larger shocks of interest in many applications. Simulations suggest that fiscal multipliers may be off by as much as 40 percent.

JEL codes: C22, C32, H20, C51, E32, E52, E60, E62

Keywords: local projection, business cycle, state-dependence, impulse response, multiplier, threshold, identification, nonlinear structural model, potential outcomes model.

[*] Acknowledgments: An earlier version of this paper circulated under the title "When do state-dependent local projections work?". We thank seminar participants at Universidade Católica Portuguesa, Erasmus School of Economics, Cambridge University, University of Glasgow, Universidad Carlos III de Madrid, Emory University, Princeton University, Yale University, University of Victoria, Simon Fraser University, University of Chicago, Toulouse School of Economics, Kansas State University, UC Riverside, and the Federal Reserve Bank of New York, as well as participants at the 3rd Italian Workshop of Econometrics and Empirical Economics, the Société Canadienne de Science Économique 2022, NASMES 2022, NBER Summer Institute 2022, NBER-NSF Time Series 2022, Women in Econometrics, VTSS 2022, MEG 2022, and ASSA 2023. We also thank Mikkel Plagborg-Møller and Rui Castro for many helpful comments and Julia Koh and Pablo Estrada for excellent research assistance. This work was supported by an NSERC grant from the government of Canada and a URC grant from Emory University. The views expressed in the paper are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of Dallas or the Federal Reserve System.

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1 Introduction

The recent empirical macroeconomics literature has emphasized the importance of allowing for nonlinearities when estimating the effects of exogenous shocks on macroeconomic variables of interest. A key question in empirical work is how impulse response functions depend on the state of the economy. For example, many studies estimating the government spending multiplier allow for the possibility that this multiplier may be different during recessions and expansions (e.g., Auerbach and Gorodnichenko (2012, 2013a,b), Bachmann and Sims (2012), Owyang, Ramey and Zubairy (2013), Caggiano, Castelnuovo, Colombo and Nodari (2015), Ramey and Zubairy (2018), Alloza (2022), and Ghassibe and Zanetti (2020)). There is also a related literature on the dependence of tax multipliers on the business cycle (e.g., Candelon and Lieb (2013), Alesina, Azzalini, Favero, Giavazzi and Miano (2018), Sims and Wolff (2018), Eskandari (2019), and Demirel (2021)). Similar questions arise in many other contexts including the analysis of monetary policy shocks. For example, Santoro, Petrella, Pfajfar and Gaffeo (2014), Tenreyro and Thwaites (2016), Angrist, Jordà and Kuersteiner (2018), Barnichon and Matthes (2018) and Klepacz (2020) allow the responses to monetary policy shocks to vary as a function of the state of the economy. Other studies allow responses to vary depending on whether the zero lower bound is binding (e.g., Auerbach and Gorodnichenko (2016); Ramey and Zubairy (2018); Mavroeidis (2021)). Yet another example of the estimation of state-dependent responses is the work of Caggiano, Castelnuovo and Groshenny (2014) who examine the dependence of the effects of uncertainty shocks on whether the economy is in recession or expansion.

Many of these studies rely on a variant of the local projection (LP) approach of Jordà (2005, 2009) (see also Dufour and Renault (1998) and Chang and Sakata (2007)) to estimate the state-dependent impulse response functions. For example, given an observed policy shock series ε_{1t}, a state-dependent local projection estimates the dynamic effect of ε_{1t} on the scalar variable y_{t+h} conditionally on the state of the economy by a set of regressions for each horizon h:

$$y_{t+h} = H_{t-1}\left[b_h(1)\,\varepsilon_{1t} + \pi'_{E,h} z_{t-1}\right] + (1 - H_{t-1})\left[b_h(0)\,\varepsilon_{1t} + \pi'_{R,h} z_{t-1}\right] + v_{t+h}, \tag{1}$$

where H_{t-1} takes the value 1 if the economy is in expansion and 0 if it is in recession. The least squares estimate $\hat{b}_h(1)$ of the slope coefficient associated with $\varepsilon_{1t} H_{t-1}$ is usually interpreted as the impulse response of y_{t+h}, conditionally on $H_{t-1} = 1$, while $\hat{b}_h(0)$ is interpreted as the response of y_{t+h} when conditioning on $H_{t-1} = 0$. The regressor z_{t-1} includes lags of all model variables.

One argument for using state-dependent local projections rather than state-dependent structural vector autoregressive (VAR) models has been their computational simplicity. Estimating impulse responses in state-dependent VAR models by numerical methods tends to be computationally more challenging than the estimation of state-dependent local projections by the method of least squares. A related argument has been that LP estimators dispense with the need to estimate equations for dependent variables other than the outcome variable of interest. Finally, unlike state-dependent VAR models, state-dependent local projections may be estimated without having to specify the process governing the transition from one state to the other. As a result, Ramey's (2016) handbook chapter concludes that "if one is interested in estimating state dependent models, the [...] local projection method is a simple way to estimate such a model and calculate impulse response functions (p. 87)".

Table 1 lists more than 50 journal articles published over the last ten years in general interest journals and field journals in macroeconomics, public economics, international economics, and applied econometrics that use this approach. State-dependent LP estimators are also discussed at length in book chapters (e.g., Auerbach and Gorodnichenko (2013b), Auerbach and Gorodnichenko (2017), Ramey (2016)) and they continue to be used extensively in recent working papers (e.g., Ahir, Bloom and Furceri (2022), Alloza, Gonzalo and Sanz (2021), Cloyne, Jordà and Taylor (2023), De Ridder, Hannon and Pfajfar (2020), Eskandari (2019), Ferriere and Navarro (2020), Gourieroux and Jasiak (2022), Jo and Zubariry (2022), Klepacz (2021), Zeev, Ramey and Zubairy (2023)).

Perhaps surprisingly, despite its widespread application, the validity of the LP approach to estimating state-dependent impulse responses has not been established to date. It has been taken as self-evident in applied work that the state-dependent LP estimator will be consistent. In this paper, we clarify the conditions under which the state-dependent LP estimator can be expected to recover the population impulse responses.[1] This task is complicated by the fact that for state-dependent processes there are alternative definitions of the population impulse response one may have in mind. For example, one possible definition of the population response is the average response of the outcome variable to a shock of magnitude δ, conditional on the state of the economy at the time when this shock occurs, building on a large literature on nonlinear impulse response analysis in time series econometrics (see, e.g., Gonçalves, Herrera,

[1] LPs have become an increasingly popular alternative to VAR based estimators of impulse responses. The original LP estimator, as discussed in Jordà (2005, 2009) and Plagborg-Møller and Wolf (2021), did not allow for the impulse response function to change depending on the state of the economy. In this paper we are not concerned with linear approximations to nonlinear processes as in Plagborg-Møller and Wolf (2021), but with approximations that are explicitly state dependent and hence nonlinear.

Kilian and Pesavento (2022)). Another possible definition is the marginal response of the outcome variable to an infinitesimal shock, conditional on the state of the economy at the time of this shock.

We formally show that, depending on how the state of the economy is determined, the state-dependent estimator may be able to recover the population response under one impulse response definition, but not under the other. This result not only affects the interpretation of the state-dependent LP estimator in applied work, but also the conditions for establishing its asymptotic validity. Hence, users need to be explicit about which population response they are interested in recovering. To the extent that this point has been discussed at all, the presumption appears to be that the state-dependent LP estimator in the limit will recover the same conditional response function as a state-dependent structural VAR estimator that allows the state of the economy to evolve. Our analysis shows that this asymptotic equivalence does not hold in general.

We find that the validity of the state-dependent estimator and its interpretation depends on whether the state of the economy evolves exogenously with respect to the economy or responds endogenously to macroeconomic shocks. In the former case, the two conditional impulse response definitions above yield the same answer (up to scale). Given that the business cycle is typically defined in terms of outcome variables such as real output or unemployment that endogenously respond to all shocks in the economy, however, this result is of limited applicability in macroeconomics. We show that, when the state of the economy is endogenous with respect to macroeconomic variables as is typically the case in applied work, much depends on the magnitude δ of the structural shock of interest. When δ is not arbitrarily close to zero, the state-dependent LP estimator will not recover the conditional average response function. However, as δ approaches zero, the state-dependent LP estimator under suitable additional assumptions will recover the conditional marginal response function. Thus, the definition of the population response of interest matters.

This raises the question of how comfortable we are with the assumption of infinitesimal δ underlying the conditional marginal responses. The answer is likely to depend on the economic context. For example, as we document, military spending news shocks in studies based on long-run quarterly data such as Ramey and Zubairy (2018) can be as large as 12 standard deviations, calling into question the use of response estimates and fiscal multipliers based on the assumption of infinitesimal δ. Even in post-war quarterly data, historical military spending news shocks have been as large as 13 standard deviations. Similarly, monetary policy shocks observed in post-war data may be as large as 11 standard deviations. Clearly, the state-dependent LP estimator will not recover the population response when δ is far from zero. For example, we present simulation evidence that fiscal multipliers in realistic settings may easily differ by as much as 40% percent from the population multiplier.

As a result, state-dependent local projections do not seem suitable for studying the propagation of large historical shocks such as the fiscal policy shocks associated with the two World Wars and the Korean War. Similar concerns also apply to more recent episodes. For example, few policymakers would consider infinitesimal fiscal policy shocks during recessions that are large enough to prompt calls for fiscal interventions in the first place such as the Great Recession. We conclude that there are many empirically plausible settings in which state-dependent LP estimators of impulse responses are not likely to provide economically meaningful estimates, also calling into question estimates of fiscal and monetary multipliers reported in the empirical literature. Our analysis, however, suggests an alternative nonparametric estimator of state-dependent responses that remains valid even when δ is far from zero, regardless of whether the state of the economy is endogenous or exogenous.

The remainder of the paper is organized as follows. In Section 2, we consider a stylized bivariate parametric model for expository purposes. This model is chosen to make the analysis as transparent as possible and to facilitate the derivation of analytical results. This section also defines the population impulse responses of interest. In Section 3, we formally derive the limit of the state-dependent LP estimator under exogenous and under endogenous states and we discuss how these results change when allowing for higher model dimensions. We show that the state-dependent LP estimator is valid when the state of the economy is exogenous regardless of the magnitude δ of the structural shock.[2] We furthermore show that this estimator remains valid under suitable assumptions when the state of the economy is endogenous and δ is infinitesimal. However, it is not valid in general when the state of the economy is endogenous and δ is nonnegligible.

These conflicting results raise the question of how large δ is in practice. In Section 4, we document that in typical applications the magnitude of δ tends to be far from zero, calling into question response derivations based on the assumption of infinitesimal δ. As a result, one would not expect the state-dependent LP estimator to be consistent in these applications.

[2]As shown in online Appendix B, this conclusion applies not only to the bivariate example in Section 2, but extends to multivariate models whether the forcing variable is i.i.d., as in our analysis in this paper, or a serially correlated exogenous process, as discussed in Alloza, Gonzalo, and Sanz (2021), or merely predetermined and endogenous.

Section 5 quantifies by simulation the asymptotic bias of the state-dependent LP estimator of the impulse responses when δ is not close to zero. We also examine a representative empirical model of macroeconomic responses to fiscal policy shocks and quantify the asymptotic bias in the cumulative fiscal multiplier often reported as a summary statistic. We find that the asymptotic biases are large enough to call into question the use of state-dependent LP estimators in this case. Section 6 suggests an alternative estimation approach. We outline a new nonparametric estimator that remains valid in applications when δ is not close to zero and could replace state-dependent LP estimators in applied work. The concluding remarks are in Section 7. The online appendix contains the proofs of the main propositions (Appendix A) and additional theoretical results for a multivariate state-dependent structural VAR model when H_t is exogenous (Appendix B). Finally, details of the simulation design and further simulation results are contained in online Appendix C and D, respectively.

2 Framework

A useful benchmark for studying the properties of state-dependent local projections is a state-dependent stationary structural VAR data generating process for $z_t \equiv (x_t, y_t)'$ that has been discussed frequently in the literature. It takes the form

$$\begin{cases} x_t = \sum_{j=1}^{p} \alpha_{j,t-1} x_{t-j} + \sum_{j=1}^{p} \delta_{j,t-1} y_{t-j} + \varepsilon_{1t}, \\ \\ y_t = \sum_{j=0}^{p} \beta_{j,t-1} x_{t-j} + \sum_{j=1}^{p} \gamma_{j,t-1} y_{t-j} + \varepsilon_{2t}, \end{cases} \tag{2}$$

where the scalar x_t is assumed to be predetermined with respect to y_t and p denotes the lag order.[3] This process includes several empirically relevant special cases. For example, often x_t is simply a directly observed exogenous shock such as a monetary policy shock or a fiscal shock ($\alpha_{j,t-1} = 0$, $\delta_{j,t-1} = 0$, $\forall j$). Alternatively, x_t may be an exogenous serially correlated process ($\delta_{j,t-1} = 0$, $\forall j$). The i.i.d. error term $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ defines the vector of mutually independent structural shocks. The variables y_t and ε_{2t} may be higher dimensional. We abstract from this possibility for now, since allowing for higher dimensions would necessitate the use of matrix notation, as in online Appendix B.

We are interested in the response over time of y_t to a one-time shock in ε_{1t} in this state-dependent structural VAR model. The variable y_t may be detrended or may be expressed in

[3] In practice, the order of the lag polynomials may differ, in which case p without loss of generality may be interpreted as the highest lag order.

growth rates, in which case the growth rate response is cumulated. These response functions may also be used to derive multipliers of interest in macroeconomics, in which case y_t includes all variables needed to define the multiplier. Identification requires the structural model to be block recursive with respect to y_t.

The model coefficients evolve over time depending on the state of the economy. Unlike in Markov switching models, the state of the economy is observed. In the simplest and most common case, there are only two states (such as recession or expansion). An important feature of the data generating process (1) is how this binary indicator is determined. For example, a recession is often defined as the unemployment rate exceeding some threshold and an expansion as the unemployment rate falling below that threshold. The unemployment rate in this example in turn may be included in y_t or not. More generally, the variable defining the state may be exogenous with respect to z_t, although that situation rarely arises in practice.

Thresholds in turn may be exogenously given or may refer to multiples of standard deviations of the variable in question from its mean over the estimation period (or standard deviations from zero, if that variable is always positive). Alternatively, the economy may be in recession if real output is below some trend line and in expansion if it is above this trend line, where the trend line may refer to a two-sided or a one-sided moving average filter or possibly some higher-order deterministic trend. More generally, the states could depend on multiple binary indicators such as whether economic uncertainty is high or low and whether the zero lower bound on the interest rate is binding or not.

2.1 A stylized structural model

To illustrate our main results, we focus on a stylized structural data generating process for $z_t = (x_t, y_t)'$ of the form

$$\begin{cases} x_t = \varepsilon_{1t} \\ y_t = \beta_{t-1} x_t + \gamma_{t-1} y_{t-1} + \varepsilon_{2t}, \end{cases} \tag{3}$$

that closely mimics key features assumed in many empirical applications. This process is a special case of data generating process (1) that facilitates the analytical derivation of the limit of the state-dependent LP estimator, allowing us to gain intuition for when the state-dependent LP estimator is expected to be valid and when it is not.

Setting $x_t = \varepsilon_{1t}$ corresponds to the empirically relevant situation of ε_{1t} being identified based on information extraneous to the model. A popular example is the narrative approach to identifying monetary policy shocks (e.g., Romer and Romer (1989), Tenreyro and Thwaites

(2016)) and fiscal policy shocks (e.g., Ramey and Shapiro (1998), Ramey (2011), Ramey (2016)).

All model coefficients evolve over time depending on the state of the economy. In the simplest case, there are only two states (such as a recession and an expansion). Accordingly, let $\beta_{t-1} = \beta_E H_{t-1} + \beta_R (1 - H_{t-1})$ and define γ_{t-1} similarly, where H_{t-1} is a binary stationary time series that takes the value 1 if the economy is in expansion and 0 otherwise. In particular, we assume that H_t is an observed (binary) deterministic function of elements of $\{w_s = (x_s, y_s, q_s)' : s \leq t\}$, a set which contains the random variables used to construct H_t. These include potentially the endogenous variables in the system $z_t = (x_t, y_t)'$ and their lags, as well as a third variable q_t (and its lags). We assume that q_t is determined outside the model and is assumed to be strictly exogenous with respect to ε_t. An example for such a variable would be a measure of "animal spirits" or "sentiment" that acts like a sunspot driving the business cycle (e.g., Blanchard (1993), Hall (1993), Barsky and Sims (2012)). For example, Blanchard (1993) suggests that the 1990-91 recession was caused by an exogenous increase in pessimism that caused a sharp reduction in aggregate demand. More specifically, let

$$H_t = \eta (w_s : s \leq t), \tag{4}$$

where $\eta(\cdot)$ is the composition of the indicator function and the function of $\{w_s : s \leq t\}$ used to indicate that H_t equals 1 or 0. For instance, if H_t is 1 whenever $y_t > 0$ and is 0 otherwise, then $H_t = \eta(y_t) \equiv 1(y_t > 0)$, in which case $w_t = y_t$.

We make the following additional assumptions.

Assumption 1 *$\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ are mutually independent structural shocks such that $\varepsilon_t \equiv (\varepsilon_{1t}, \varepsilon_{2t})' \sim i.i.d.(0, \Sigma)$, where Σ is a diagonal matrix with diagonal elements given by σ_i^2 for $i = 1, 2$. In addition, y_t is strictly stationary and ergodic.*

Assumption 2 *$\{q_t\}$ is independent of $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$.*

Assumption 1 is stricter than a martingale difference sequence assumption on ε_t and rules out conditional heteroskedasticity, but is standard in the nonlinear structural VAR literature. In addition, we assume that the process for y_t is strictly stationary and ergodic. Assumption 2 formalizes the idea that q_t is a strictly exogenous process that is determined outside the economic system. Note that this assumption does not rule out temporal dependence in q_t. For instance, if q_t captures exogenous consumer or investor sentiment, we are not assuming that q_t is i.i.d. However, we are assuming that it is strictly exogenous with respect to model variables such as inflation or output. The main reason we introduce this exogenous random variable q_t is because the limit of the state-dependent LP estimator depends on whether H_t is endogenous or exogenous. The latter case corresponds to setting η as a function of q_t (and its lagged values) only.

Next, we introduce two possible definitions of the impulse response function conditional on the state of the economy.

2.2 Conditional impulse response functions

Our goal is to define the causal effect on y_{t+h} of a one-time shock in ε_{1t}, conditionally on H_{t-1}, the state of the economy at time $t-1$. The latter conditioning set has been standard in the literature on state-dependent LP regressions since Auerbach and Gorodnichenko (2013a,b), who in turn build on the assumptions in the state-dependent structural VAR model in Auerbach and Gorodnichenko (2012). The state dependence of the population process has implications for the definition of the conditional impulse response function. A common approach inspired by the literature on nonlinear impulse response functions (e.g., Gallant, Rossi and Tauchen (1993), Koop, Pesaran and Potter (1996), Potter (2000), Gourieroux and Jasiak (2005), Kilian and Vigfusson (2011), Gonçalves, Herrera, Kilian and Pesavento (2021, 2022)) is to compare, all else equal, two sample paths for the outcome variable of interest, one where ε_{1t} is subject to a one-time shock at time t and another one where no such shock is present. We follow this approach here, but formalize it using a potential outcomes framework. Although the latter approach is common in the microeconometric literature on treatment effects, it only gradually has gained traction in macroeconometrics (e.g., White (2006), White and Kennedy (2009), Angrist and Kuersteiner (2011), Angrist, Jordà, and Kuersteiner (2018), and most recently Rambachan and Shephard (2021) and Cloyne, Jordà and Taylor (2023)).

A potential outcome model is a model that tells us the observed value of y_{t+h} for any fixed value of ε_{1t}. To distinguish between the random variable ε_{1t} and any fixed value it might take, we denote the latter by e. Thus, if ε_{1t} takes on values in a set A, then $e \in A$. For instance, if ε_{1t} is a binary treatment, we have only two possible values for e, 0 and 1, in which case $A = \{0, 1\}$. In the macroeconomic setting considered here, ε_{1t} is a continuous random variable.

Let $y_{t+h}(e)$ define the potential outcome associated with fixing ε_{1t} at any possible value e in the support of ε_{1t}. When $e = \varepsilon_{1t}$, we obtain the observed value of y_{t+h}, i.e., $y_{t+h}(\varepsilon_{1t}) = y_{t+h}$. Our definition of the conditional impulse response function is based on comparing this baseline value with $y_{t+h}(\varepsilon_{1t} + \delta)$, the counterfactual value of y at $t + h$ that would have been observed if ε_{1t} had been subject to a shock of discrete size δ (see, e.g., Potter (2000)).

Definition 1 (Conditional average response) *The conditional average response function of y_{t+h} to a one-time shock of fixed size δ in ε_{1t} is defined as*

$$CAR_h\left(\delta, \bar{h}\right) = E\left(y_{t+h}\left(\varepsilon_{1t} + \delta\right) - y_{t+h}\left(\varepsilon_{1t}\right) | H_{t-1} = \bar{h}\right)$$

where $\bar{h} \in \{0, 1\}$.

In our setup, the potential outcome $y_{t+h}(e)$ is a random variable obtained by solving the structural model (3) and (4) forward, letting $\varepsilon_{1t} = e$. This defines $y_{t+h}(e) = m_h(e, U_{t+h})$, where m_h is a potentially complicated function of e and U_{t+h}, and U_{t+h} contains the structural shocks on the two variables up to time $t+h$, except for ε_{1t} which is set at e, as well as the values of q between $t-1$ and $t+h-1$, and the initial condition z_{t-1}. The conditional average response function (CAR) is the expectation of $y_{t+h}(\varepsilon_{1t} + \delta) - y_{t+h}(\varepsilon_{1t})$ with respect to $(\varepsilon_{1t}, U_{t+h})$, conditionally on H_{t-1}. We call this object the conditional average response function by analogy to the notion of a (conditional) average treatment effect.

Definition 1 is similar but not identical to the definition of a conditional nonlinear response function in Gallant, Rossi and Tauchen (1993), Koop, Pesaran and Potter (1996), Potter (2000), Gonçalves, Herrera, Kilian and Pesavento (2021), and Rambachan and Shephard (2021), among others. One difference is that earlier studies conditioned on the entire information set known at time $t-1$ (i.e. the filtration $\mathcal{F}^{t-1}$ generated by the past of (y_t, x_t, q_t) in our framework). Instead, we condition on a much smaller information set, consisting of H_{t-1} only. The reason for conditioning on H_{t-1} only and not on $\mathcal{F}^{t-1}$ is that this corresponds to empirical practice. Notably, most applied researchers are interested in the impulse response function of an outcome variable conditional on being in an expansion or in a recession. The second difference is that some studies compare two potential outcomes: $y_{t+h}(e)$ and $y_{t+h}(e')$, where e and e' are fixed. We instead are comparing $y_{t+h}(\varepsilon_{1t} + \delta)$ against $y_{t+h}(\varepsilon_{1t})$, where ε_{1t} is the actual random variable that has generated the data y_{t+h}. Thus, we are comparing the observed value $y_{t+h} = y_{t+h}(\varepsilon_{1t})$ against a counterfactual value $y_{t+h}(\varepsilon_{1t} + \delta)$ that is not observed. Since ε_{1t} is random, the conditional expectation in Definition 1 averages over all possible realizations of ε_{1t} (in addition to the other sources of randomness that enter into the potential outcomes), conditionally on H_{t-1}.

Another possible definition of a conditional impulse response function is one where the size of the shock is infinitesimal. This corresponds to a conditional marginal treatment effect in microeconometrics and for this reason we will call it a "conditional marginal response" (CMR). In the following, for any function $m(e)$, we use $m'(e)$ to denote $\frac{\partial}{\partial e}m(e)$.

Definition 2 (Conditional marginal response) *The conditional marginal response function of y_{t+h} to an infinitesimal shock in ε_{1t} is defined as*

$$CMR_h\left(\bar{h}\right) = E\left(y'_{t+h}\left(\varepsilon_{1t}\right)|H_{t-1} = \bar{h}\right),$$

for any value of $\bar{h} \in \{0, 1\}$.

Definition 2 corresponds to the derivative of the nonlinear impulse response function defined in Potter (2000, p. 1431), if we do not condition on H_{t-1}. Definition 2 is also a version of Definition 3 in Rambachan and Shephard (2021). Specifically, it corresponds to their definition of the marginal filtered treatment effect $E\left(y'_{t+h}\left(e\right)|\mathcal{F}^{t-1}\right)$ when we replace e with ε_{1t} and $\mathcal{F}^{t-1}$ by H_{t-1}.

Although considering the effect of an infinitesimal shock on an outcome variable of interest is not as popular in macroeconomics as considering the effect of a shock of fixed magnitude δ, we consider both the conditional marginal and conditional average response functions because local projection estimands may relate to either of these two definitions. In particular, as we will show next, state-dependent local projections identify the conditional marginal response function when H_t is exogenous under Assumptions 1 and 2. Under these assumptions, the LP estimand is also equal to the conditional average response function for $\delta = 1$. In contrast, when H_t is endogenous, the state-dependent LP estimator is inconsistent for the conditional average response function, although it may still recover the conditional marginal response function under additional assumptions (see Assumption 3 below).

Before we proceed, note that the conditional marginal response function can be written as the limit of the ratio of the conditional average response function and the size of the shock δ, as $\delta \to 0$, when ε_{1t} and U_{t+h} are conditionally independent. This conditional independence assumption holds under our assumptions, as proved in Lemma A.1 in online Appendix A. We state this result next, assuming that the conditions required to interchange the order of the integration and the limit apply.

Lemma 2.1 *Under the structural model (3) and (4), assuming that Assumptions 1 and 2 hold,*

$$\lim_{\delta \to 0} \frac{CAR_h\left(\delta, \bar{h}\right)}{\delta} = CMR_h\left(\bar{h}\right)$$

for any $\bar{h} \in \{0, 1\}$.

3 State-dependent local projections

Recall that a state-dependent local projection estimates the dynamic effect of ε_{1t} on y_{t+h} conditionally on the state of the economy by a set of regressions for each horizon h:

$$y_{t+h} = H_{t-1}\left[b_h(1)\,\varepsilon_{1t} + \pi'_{E,h} z_{t-1}\right] + (1 - H_{t-1})\left[b_h(0)\,\varepsilon_{1t} + \pi'_{R,h} z_{t-1}\right] + v_{t+h}, \tag{5}$$

where H_{t-1} takes the value 1 if the economy is in expansion and 0 if it is in recession. The least squares estimate $\hat{b}_h(1)$ of the slope coefficient associated with $\varepsilon_{1t} H_{t-1}$ is usually interpreted as the impulse response of y_{t+h}, conditionally on $H_{t-1} = 1$, whereas $\hat{b}_h(0)$ is interpreted as the response of y_{t+h} when conditioning on $H_{t-1} = 0$. The main goal of this section is to clarify the interpretation of this LP estimator and the conditions required for a causal interpretation of the estimates. Specifically, we derive the probability limits of $\hat{b}_h(1)$ and $\hat{b}_h(0)$ and relate these estimands to the two definitions of the conditional impulse response function given above. We derive results for two scenarios: one in which H_t is exogenous and another one in which H_t is endogenous with respect to ε_t.

3.1 Exogenous H_t

We assume that (4) holds with $w_s = q_s$ such that $H_t = \eta(q_s : s \leq t)$. This corresponds to the case of H_t being constructed using only information on the variable q_t. Given Assumption 2, this implies that H_t is exogenous with respect to ε_{1t} and ε_{2t}.

We first derive the conditional impulse response functions $CAR_h\left(\delta, \bar{h}\right)$ and $CMR_h\left(\bar{h}\right)$ for this model. Under our assumptions, the potential outcome model $y_{t+h}(e)$ is

$$y_{t+h}(e) = (\gamma_{t+h-1} \cdots \gamma_t)\,\beta_{t-1} e + V_{t+h} \equiv m_h(e, U_{t+h}), \tag{6}$$

where $(\gamma_{t+h-1} \cdots \gamma_t)$ and V_{t+h} are both functions of $U_{t+h} \equiv (\varepsilon_{t+h}, \ldots, \varepsilon_{t+1}, \varepsilon_{2t}, q_{t+h-1}, \ldots, q_{t-1}, z_{t-1})'$. This follows easily by iterating on the model equation for y_t in (3) fixing $\varepsilon_{1t} = e$ and using the fact that H_t is exogenous (and, hence, not affected by e). Thus, for any e,

$$y_{t+h}(e + \delta) - y_{t+h}(e) = \left[(\gamma_{t+h-1} \cdots \gamma_t)\,\beta_{t-1}\right]\delta,$$

and

$$CAR_h\left(\delta,\bar{h}\right) = E\left[\left(\gamma_{t+h-1}\cdots\gamma_t\right)\beta_{t-1}|H_{t-1}=\bar{h}\right]\delta.$$

Next we use (6) to obtain the conditional marginal response function for this model. Since m_h is a linear function of e, it follows that

$$y'_{t+h}\left(e\right) \equiv \frac{\partial}{\partial e} m_h\left(e, U_{t+h}\right) = \left(\gamma_{t+h-1}\cdots\gamma_t\right)\beta_{t-1}.$$

This implies that

$$CMR_h\left(\bar{h}\right) = E\left[\left(\gamma_{t+h-1}\cdots\gamma_t\right)\beta_{t-1}|H_{t-1}=\bar{h}\right] = CAR_h\left(1,\bar{h}\right),$$

and shows that the conditional marginal response function coincides with the conditional average response function $CAR_h\left(\delta,\bar{h}\right)$ for a shock of size $\delta = 1$.

We summarize these results in the next proposition. Let $\beta_{\bar{h}} = \beta_E$ if $\bar{h} = 1$ and $\beta_{\bar{h}} = \beta_R$ if $\bar{h} = 0$.

Proposition 3.1 *Assume the structural process is (3) and (4) with $H_t = \eta\left(q_s : s \le t\right)$. Under Assumptions 1 and 2, the following results hold for $\bar{h} \in \{0,1\}$:*

(i) *For any fixed δ, $CAR_0\left(\delta,\bar{h}\right) = \beta_{\bar{h}}\delta$, and for any $h \ge 1$,*

$$CAR_h\left(\delta,\bar{h}\right) = E\left[\gamma_{t+h-1}\cdots\gamma_t|H_{t-1}=\bar{h}\right]\beta_{\bar{h}}\delta.$$

(ii) *$CMR_0\left(\bar{h}\right) = \beta_{\bar{h}}$, and for any $h \ge 1$,*

$$CMR_h\left(\bar{h}\right) = E\left[\gamma_{t+h-1}\cdots\gamma_t|H_{t-1}=\bar{h}\right]\beta_{\bar{h}}.$$

Part (i) of Proposition 3.1 gives the conditional average response function for any shock of size δ, where δ is fixed. For $h = 0$, the impact response function of y_t to a shock of size $\delta = 1$ in ε_{1t} is $\beta_{\bar{h}}$, which is either β_E or β_R depending on whether we were in an expansion ($\bar{h} = 1$) or in a recession ($\bar{h} = 0$) prior to the shock. For longer horizons, the conditional average response function depends on the state of the economy at time $t-1$, but not on the current or future states of the economy. Nor do we condition on the history of states prior to $t-1$. Rather, we average them out and condition only on the most recent state. This corresponds to the standard approach in estimating state-dependent responses in applied macroeconomics, when interest centers on the question of how the impulse response function differs, depending on whether the economy was in expansion or recession prior to the shock.

Part (ii) of Proposition 3.1 gives the conditional marginal response function in Definition 2. This response function traces the dynamic causal effect of an infinitesimal shock in ε_{1t} on y_{t+h}. Since $CMR_h\left(\bar{h}\right) = \lim_{\delta\to 0} CAR_h\left(\delta,\bar{h}\right)/\delta$ (as stated in Lemma 2.1), part (ii) follows immediately from part (i) after dividing $CAR_h\left(\delta,\bar{h}\right)$ by δ. Thus, if H_t is exogenous, the conditional average response function for a shock of magnitude δ in proportion to δ equals the conditional marginal response function. This is true for any fixed δ and hence also for $\delta \to 0$.

Next, we derive the probability limits of the state-dependent LP estimates $\hat{b}_h(1)$ and $\hat{b}_h(0)$. We can obtain each of these separately, by restricting the sample to $H_{t-1} = 1$ and $H_{t-1} = 0$, respectively. For instance, $\hat{b}_h(1)$ can be obtained by a regression of y_{t+h} on $\varepsilon_{1t}H_{t-1}$ and $z_{t-1}H_{t-1}$ (omitting $\varepsilon_{1t}(1-H_{t-1})$ and $z_{t-1}(1-H_{t-1})$ from the regression). This follows because $H_{t-1}(1-H_{t-1}) = 0$ for all t.

Under the assumed stationarity and ergodicity of ε_{1t} and y_t, it can be shown easily that

$$\hat{b}_h\left(\bar{h}\right) \to_p b_h\left(\bar{h}\right) = \frac{E\left(y_{t+h}\varepsilon_{1t}|H_{t-1}=\bar{h}\right)}{E\left(\varepsilon_{1t}^2|H_{t-1}=\bar{h}\right)},$$

where the LP estimand $b_h\left(\bar{h}\right)$ can be interpreted as the population OLS coefficient associated with ε_{1t} in a linear regression of y_{t+h} on ε_{1t} which conditions on $H_{t-1} = \bar{h}$.

Proposition 3.2 *Consider the structural process (3) and (4) with $H_t = \eta\left(q_s : s \le t\right)$. If Assumptions 1 and 2 hold, then for $\bar{h} \in \{0,1\}$,*

$$b_h\left(\bar{h}\right) = CMR_h\left(\bar{h}\right) = \frac{CAR_h\left(\delta,\bar{h}\right)}{\delta} = CAR_h\left(1,\bar{h}\right).$$

The main implication of Proposition 3.2 is that state-dependent local projections "work" when H_t is strictly exogenous with respect to the structural shocks in the model, as would be the case when q_t represents an exogenous measure of "sentiment". Under these conditions, $b_h\left(\bar{h}\right)$ is equal to the conditional marginal response function, which gives the effect of an infinitesimal size shock in ε_{1t} on y_{t+h}. However, Proposition 3.2 also shows that another valid interpretation of $b_h\left(\bar{h}\right)$ is that it gives the conditional average response of y_{t+h} to a shock of fixed size δ, in proportion to δ. When $\delta = 1$, $b_h\left(\bar{h}\right)$ captures the conditional average effect of a shock of size 1 in ε_{1t} on y_{t+h}. Both interpretations are correct and coincide with each other under the assumed exogeneity of H_t.

The equation for y_{t+h} helps illustrate why the exogeneity of H_t is important for deriving this result. To see this, condition on $H_{t-1} = 1$ being an expansion such that $\bar{h} = 1$. At horizon $h = 0$, the model implies that

$$y_t = \beta_E \varepsilon_{1t} + \underbrace{\gamma_E y_{t-1} + \varepsilon_{2t}}_{=v_t},$$

where the underlying error term v_t is independent of ε_{1t} under Assumptions 1 and 2, conditionally on $H_{t-1} = 1$. Thus, the state-dependent LP estimand is $b_0(1) = \beta_E$ and the LP regression recovers both of the conditional average and marginal impulse response functions on impact.

For horizon $h = 1$, conditionally on $H_{t-1} = 1$, the equation for y_{t+1} is

$$y_{t+1} = \beta_t \varepsilon_{1t+1} + \gamma_t (\beta_E \varepsilon_{1t} + \gamma_E y_{t-1} + \varepsilon_{2t}) + \varepsilon_{2t+1} = \gamma_t \beta_E \varepsilon_{1t} + v_{t+1},$$

where $v_{t+1} = \gamma_t \gamma_E y_{t-1} + \gamma_t \varepsilon_{2t} + \beta_t \varepsilon_{1t+1} + \varepsilon_{2t+1}$. This model has a heterogeneous slope coefficient $\gamma_t \beta_E$ because γ_t is a function of the state indicator H_t. The regression of y_{t+1} on ε_{1t} recovers the conditional expectation of $\gamma_t \beta_E$, conditionally on $H_{t-1} = 1$, provided v_{t+1} is conditionally independent of ε_{1t}. Setting $H_t = \eta(q_s : s \leq t)$ with q_s satisfying Assumption 2, the LP estimand for $h = 1$ reduces to

$$b_1(1) = E(\gamma_t | H_{t-1} = 1) \beta_E.$$

For general values of h, we can write y_{t+h} as a function of ε_{1t} and an error term that depends on $H_{t+h-1}, \ldots, H_{t-1}$. Conditionally on H_{t-1}, this equation is state-dependent, as it depends on $H_{t+h-1}, \ldots, H_t$. A linear local projection of y_{t+h} on ε_{1t} which conditions only on $H_{t-1} = 1$ recovers the conditional average and marginal response functions provided the error term is orthogonal to ε_{1t}, conditionally on H_{t-1}. Since this error depends on $H_{t+h-1}, \ldots, H_t$, we require that ε_{1t} be independent of $H_{t+h-1}, \ldots, H_t$, conditionally on H_{t-1}. This independence condition holds under the assumption that $H_t = \eta(q_s : s \leq t)$ and the process q_t is independent of ε_t, as assumed in Assumption 2.[4]

While this result was derived in the simplest possible setting, in online Appendix C we show that the validity of the state-dependent LP estimator, when H_t is exogenous generalizes to a multivariate state-dependent structural VAR process for $z_t = (x_t, y_t')'$, where y_t is an $n \times 1$ vector of endogenous variables and x_t is predetermined with respect to y_t.

[4]A milder sufficient assumption is that the conditional first two moments of ε_t are independent of $H_{t+h-1}, \ldots, H_t$, conditionally on the available information at time $t - 1$.

3.2 Endogenous H_t

We now characterize the state-dependent LP estimand when H_t is endogenous with respect to the structural shocks of the system ε_t. To facilitate the derivation of analytical results, we add the following assumption:

Assumption 3 *(a)* $H_t = \eta(\varepsilon_{1t}) = 1(\varepsilon_{1t} > c)$, *where* c *is any constant.* *(b)* $\varepsilon_{1t} \sim$ *i.i.d.* $N(0, \sigma_1^2)$.

Although assuming that H_t is a function of the outcome variable y_t would be more realistic, we focus on a simpler process in which H_t depends on ε_{1t} because this greatly simplifies the mathematical derivations.[5]

In order to derive the conditional (average and marginal) impulse response functions in this model, we first derive the potential outcomes $y_{t+h}(e)$. As previously, they are obtained by iterating forward the equation for y_t given in (3), fixing $\varepsilon_{1t} = e$. In contrast to the earlier case in which H_t was exogenous, the endogeneity of H_t creates a nonlinearity in the function m_h that defines $y_{t+h}(e) = m_h(e, U_{t+h})$. In particular, for $h = 0$, $y_t(e) = \beta_{t-1}e + V_{0t}$, where $V_{0t} = \gamma_{t-1}y_{t-1} + \varepsilon_{2t}$, a function of $U_t = (\varepsilon_{2t}, z_{t-1})$. This is still a linear function of e, as was the case for H_t exogenous. However, for horizon $h = 1$, we now obtain that

$$y_{t+1}(e) = \gamma(e)\,\beta_{t-1}e + V_{t+1}(e) \equiv m_1(e, U_{t+1}),$$

where $V_{t+1}(e) = \gamma(e)V_{0t} + \beta(e)\varepsilon_{1t+1} + \varepsilon_{2t+1} \equiv V_1(e, U_{t+1})$, is a nonlinear function of e and $U_{t+1} = (\varepsilon_{t+1}, \varepsilon_{2t}, z_{t-1})'$. More generally, for any $h > 1$, it can be shown that

$$y_{t+h}(e) = [\gamma_{t+h-1} \cdots \gamma_{t+1}\gamma(e)]\,\beta_{t-1}e + V_{t+h}(e) \equiv m_h(e, U_{t+h}), \tag{7}$$

where $V_{t+h}(e)$ is a nonlinear function of e and $U_{t+h} \equiv (\varepsilon_{t+h}, \ldots, \varepsilon_{t+1}, \varepsilon_{2t}, z_{t-1})'$. There are two important differences from the potential outcome model derived under the assumption of exogenous H_t (see eq. (6)). First, since $\gamma(e) = \gamma_R + (\gamma_E - \gamma_R)\,\eta(e)$, where $\eta(e) = 1(e > c)$ is a nonlinear function of e, the first term of (7) is nonlinear in e. Second, as shown in online Appendix A, the term $V_{t+h}(e)$ is also a nonlinear function of e (whereas V_{t+h} did not depend on e in (6)). This makes $y_{t+h}(e)$ a nonlinear function of e.

[5] In particular, under Assumption 3(a) H_t is i.i.d. since ε_{1t} is i.i.d. This implies that a shock on ε_{1t} only impacts the date t coefficients in the model for y_t. Nevertheless, all the conditional impulse response functions are affected by this shock. The Gaussianity assumption on ε_{1t} is also instrumental to obtain the closed form expression for the local projection estimands in this setting.

Using the potential outcomes (7), we obtain the following result. As before, let $\beta_{\bar{h}} = \beta_E$ if $\bar{h} = 1$ and $\beta_{\bar{h}} = \beta_R$ if $\bar{h} = 0$, and define $\gamma_{\bar{h}}$ similarly. Also let $\bar{\gamma} \equiv E(\gamma_t) = \gamma_R + (\gamma_E - \gamma_R)\,\Phi(-c/\sigma_1)$ and $v_{\bar{h}} \equiv \gamma_{\bar{h}} E\left(y_{t-1}|H_{t-1} = \bar{h}\right)$, where we let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cumulative density function (cdf) and probability density function (pdf) of a standard normal distribution.

Proposition 3.3 *Assume the structural process is (3) and (4) with $H_t = \eta(\varepsilon_{1t})$. Under Assumptions 1 and 3, we have that for $\bar{h} \in \{0,1\}$:*

(i) $CAR_0\left(1,\bar{h}\right) = \beta_{\bar{h}}$, *and for $h \geq 1$, $CAR_h\left(1,\bar{h}\right) = (\bar{\gamma})^{h-1} CAR_1\left(\delta,\bar{h}\right)$, where*

$$
\begin{aligned}
CAR_1\left(\delta,\bar{h}\right) &= \{\gamma_R + (\gamma_E - \gamma_R)\,\Phi(-c/\sigma_1)\}\beta_{\bar{h}}\delta \\
&\quad + \{\gamma_R + (\gamma_E - \gamma_R)\,[\Phi(-c/\sigma_1 + \delta/\sigma_1) - \Phi(-c/\sigma_1)]\}\beta_{\bar{h}}\delta \\
&\quad + \{(\gamma_E - \gamma_R)\,\sigma_1[\phi(-c/\sigma_1 + \delta/\sigma_1) - \phi(-c/\sigma_1)]\}\beta_{\bar{h}} \\
&\quad + \{(\gamma_E - \gamma_R)\,[\Phi(-c/\sigma_1 + \delta/\sigma_1) - \Phi(-c/\sigma_1)]\}v_{\bar{h}}.
\end{aligned}
$$

(ii) $CMR_0\left(1,\bar{h}\right) = \beta_{\bar{h}}$, *and for any $h \geq 1$, $CMR_h\left(\bar{h}\right) = (\bar{\gamma})^{h-1} CMR_1\left(\bar{h}\right)$, where*

$$
CMR_1\left(\bar{h}\right) = \{\gamma_R + (\gamma_E - \gamma_R)\,\Phi(-c/\sigma_1)\}\beta_{\bar{h}} + \{(\gamma_E - \gamma_R)\,\sigma_1^{-1}\phi(c/\sigma_1)\}\left(c\beta_{\bar{h}} + v_{\bar{h}}\right).
$$

As the proof of Proposition 3.3 in the online Appendix A reveals, we can decompose $CAR_1\left(\delta,\bar{h}\right)$ into the sum of a direct effect and an indirect effect given by

$$
\begin{aligned}
\text{Direct effect} &= E(\gamma(\varepsilon_{1t}))\beta_{\bar{h}}\delta \\
\text{Indirect effect} &= E\left[(\gamma(\varepsilon_{1t} + \delta) - \gamma(\varepsilon_{1t}))\right]\left(\beta_{\bar{h}}\delta + v_{\bar{h}}\right) + E\left[(\gamma(\varepsilon_{1t} + \delta) - \gamma(\varepsilon_{1t}))\,\varepsilon_{1t}\right]\beta_{\bar{h}}.
\end{aligned}
$$

The direct effect can also be written as $E(\gamma_t)\,\beta_{\bar{h}}\delta$ (since $\gamma(\varepsilon_{1t}) = \gamma_t$), which coincides with what was derived in Proposition 3.1 for exogenous H_t. It captures the effect of a change in ε_{1t} on y_{t+h} that keeps γ_t constant, as would have been the case if H_t had been exogenous. However, in the current model, $H_t = \eta(\varepsilon_{1t})$. Thus, perturbing ε_{1t} by δ also has an impact on the model parameters at time t. This indirect effect depends on the difference between $\gamma(\varepsilon_{1t} + \delta)$ and $\gamma(\varepsilon_{1t})$. The expressions provided in Proposition 3.3 are the result of evaluating these expectations under the Gaussianity assumption on ε_{1t}. Note that part (ii) of Proposition 3.3 follows from part (i) by considering the limit of $\delta^{-1} CAR_h\left(\delta,\bar{h}\right)$ as $\delta \to 0$. More generally, Proposition 3.3 shows that the conditional average response function no longer coincides with the conditional marginal response function when H_t is endogenous.

Next, we derive the state-dependent local projection estimands $b_h\left(\bar{h}\right)$ and show that they coincide with $CMR_h\left(\bar{h}\right)$.

Proposition 3.4 *Assume the structural model is (3) and (4) where $H_t = \eta\left(\varepsilon_{1t}\right)$. Under Assumptions 1 and 3, we have that for $\bar{h} \in \{0, 1\}$, for any $h \geq 0$, $b_h\left(\bar{h}\right) = CMR_h\left(\bar{h}\right)$.*

Given Proposition 3.3, Proposition 3.4 has two main implications. First, the state-dependent LP estimator is consistent for the conditional marginal response function at all horizons $h \geq 0$. This holds under Assumptions 1 and 3, which allows H_t to be endogenous of the form $H_t = 1\left(\varepsilon_{1t} > c\right)$, where $\varepsilon_{1t} \sim N\left(0, \sigma_1^2\right)$. Thus, the state-dependent LP estimand can be interpreted as giving the effect of an infinitesimally sized shock ε_{1t} on y_{t+h} under these simplified assumptions. The second implication is that, in general, the state-dependent LP does not recover the conditional average response function. In particular, the asymptotic bias of LP, when the target impulse response function is given by the conditional average response for $\delta = 1$, is the difference between $CMR_h\left(\bar{h}\right)$ in part (b) of Proposition 3.3 and $CAR_h\left(1, \bar{h}\right)$. More generally, LP does not consistently estimate the dynamic causal effect of a perturbation by a fixed δ in ε_{1t} on y_{t+h}.

Assumption 3 is a simplifying assumption that allowed us to obtain the analytical results in Propositions 3.3 and 3.4. An alternative approach would have been to obtain the state-dependent LP estimand $b_h\left(\bar{h}\right)$ using high-level assumptions on the potential outcomes, as in Rambachan and Shephard (2021) (see the assumptions underlying their Theorem 3, for example). We conjecture that, under such assumptions, $b_h\left(\bar{h}\right)$ could be written as a weighted average of $E\left(y'_{t+h}\left(e\right) | H_{t-1} = \bar{h}\right)$ with non-negative weights $\omega\left(e\right)$ that integrate to 1. We do not pursue this approach here for two reasons. One reason is that specifying a class of parametric models that is well accepted in the literature makes it easier to relate our analysis to existing work. Second, specifying (3) and imposing Assumptions 1 and 3 allows us to show that $b_h\left(\bar{h}\right)$ is equal to the conditional marginal response function $CMR_h\left(\bar{h}\right)$. We believe the former is easier to interpret for applied users than a weighted average of marginal response functions would have been.

Our proof of the invalidity of the LP estimator when H_t is endogenous and δ is not close to zero clearly illustrates the essence of the problem in the context of a stylized model. The analysis of more general models tends to become analytically intractable when H_t is endogenous.

However, as shown in Sections 4 and 5, simulation evidence for such models tends to support the results we analytically derived for our simpler model.

4 Which asymptotic result is more relevant for applied work?

When H_t is endogenous, as is typically the case in practice, the researcher needs to decide whether the object of interest is the conditional average response or the conditional marginal response. If the latter is the object of interest, state-dependent LP estimators provide an easy and consistent way of estimating this impulse response. In contrast, if the researcher is interested in estimating the effect of a shock of fixed size δ, consistent estimation of the conditional average response is required. Ultimately, this choice should be guided by the research question the practitioner is interested in.

4.1 Evidence on the magnitude of shocks

For example, consider the response of government spending and GDP to the quarterly military spending news shock studied in Ramey (2011) and Ramey and Zubairy (2018).[6] The data span the period from 1890Q1 to 2015Q4. The standard deviation of this shock series is 0.0597. The maximum value is 0.692 and is observed in 1941Q4. This realization amounts to almost twelve times the standard deviation of the shock series. Thus, if we interpret x_t as an i.i.d. shock series, as Ramey and Zubairy did, World War II was associated with a military spending news shocks of 12 standard deviations, calling into question response estimates based on the premise of an infinitesimal δ. The shock associated with World War I was only slightly smaller. Even in post-war quarterly data, there are large shocks. For example, the shock associated with the Korean War in 1950Q3 was about thirteen times this subsample's standard deviation. Thus, given the results in Section 3, state-dependent local projections should not be used to study the propagation of the large historical shocks such as the fiscal policy shocks associated with the two World Wars and the Korean War.

What about the analysis of fiscal interventions in more recent decades? While military buildups of the size observed during these three wars are rare, military spending shocks that

[6] The defense spending news series originally constructed by Ramey (2011) and then expanded by Ramey and Zubairy(2018) is defined as the change in the expected present discounted value of government spending for events that were related to political or military events. The nominal value of these changes are then divided by nominal GDP lagged one quarter.

exceed one standard deviation were quite common even in recent decades. This is also true for fiscal spending shocks more broadly. An obvious concern is that countercyclical fiscal policy is explicitly designed to move the economy from a recession to an expansion suggesting that the indirect effect of fiscal policy shocks matters. Indeed, few policymakers would consider infinitesimal fiscal policy interventions during recessions that are large enough to prompt calls for fiscal action such as the Great Recession. Thus, fiscal policy analysis almost invariably is about large fiscal policy shocks the dynamic effects of which state-dependent local projections are unlikely to capture.

Nor are large shocks uncommon in other applications. A case in point is the narrative monetary policy shock series of Romer and Romer (2004), as updated by Wieland and Yang (2020). Given data from January 1969 to December 2007, the monthly series of monetary policy shocks has a standard deviation of $\sigma = 0.296$. Policy shocks range from -3.25 in April 1980 to 1.86 in November 1980, which implies shocks with magnitudes of between -11 and 6 standard deviations. After aggregating these data to quarterly frequency, the magnitude of the policy shocks ranges from -7 to 4 standard deviation. More generally, monetary policy shocks that exceed one standard deviation are common.

4.2 How much does the choice of δ matter?

This evidence motivates a closer look at the mechanism that makes state-dependent local projections fail when δ is large and H_t is endogenous. Intuitively, given the analysis in Section 3, one would expect the LP estimator to provide a good approximation as long as the economy remains in the same state following a shock. All else equal, this is more likely to be the case when δ is close to zero. Figure 1 illustrates this point for $\delta \in \{0.25, 1, 5\}$. It shows the asymptotic bias of the impulse responses obtained by state-dependent local projections (expressed in percent deviations from the population response). For illustrative purposes, all results are generated from the data generating process (DGP)

$$\begin{aligned} x_t &= \rho x_{t-1} + \varepsilon_{1t} \\ y_t &= \beta_{t-1} x_t + \alpha_{t-1} x_{t-1} + \gamma_{t-1} y_{t-1} + \varepsilon_{2t}, \end{aligned} \tag{8}$$

where

$$
\begin{aligned}
\alpha_{t-1} &= \alpha_E H_{t-1} + \alpha_R(1 - H_{t-1}) \\
\beta_{t-1} &= \beta_E H_{t-1} + \beta_R(1 - H_{t-1}), \\
\gamma_{t-1} &= \gamma_E H_{t-1} + \gamma_R(1 - H_{t-1}),
\end{aligned} \tag{9}
$$

$\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})' \sim N(0, I_2)$ and H_t is an indicator function for the state of the economy. When $H_t = 1$, the economy is in expansion, and, when $H_t = 0$, the economy is in recession. The state of the economy is endogenously determined as $H_t = 1(y_t > 0)$. We consider two special cases of this process. DGP 1 sets $\rho = 0$ such that x_t is a directly observed i.i.d. shock, as is often the case in applied work. Furthermore, we set $\beta_E = 2.5$, $\beta_R = 3.5$, $\gamma_E = 0.7$, $\gamma_R = 0.1$ and $\alpha_{t-1} = 0$. In DGP 2, x_t follows an AR(1) process with $\rho = 0.8$, motivated by the analysis in Alloza, Gonzalo and Sanz (2021), and $\alpha_{t-1} \neq 0$, given $\alpha_E = 1.2$ and $\alpha_R = 0.9$. The population response is evaluated as $CAR_h\left(\delta, \overline{h}\right) = E\left(y_{t+h}\left(\varepsilon_{1t} + \delta\right) - y_{t+h}\left(\varepsilon_{1t}\right)|H_{t-1}\right)$, whereas the LP estimands are evaluated as $b_h(1) = \frac{E(x_t H_{t-1} y_{t+h})}{E\left(x_t^2 H_{t-1}\right)}$ and $b_h(0) = \frac{E(x_t(1-H_{t-1})y_{t+h})}{E\left(x_t^2(1-H_{t-1})\right)}$.[7] The number of draws used to compute these conditional expectations is 50 million, which ensures that the LP impact response matches the population response.

Figure 1 confirms that the size of the asymptotic bias at longer horizons declines as $\delta \to 0$, but can be substantial for large δ, as predicted by Propositions 3.3 and 3.4. The bias can be as large as 82 percent for $\delta = 5$ and as large as 23 percent for $\delta = 0.25$. This result is qualitatively robust to the use of alternative DGPs.

Further insight may be gained by decomposing $CAR\left(\delta, \overline{h}\right)$ into the direct effect of the shock and the indirect effect (obtained as the difference between the population response and the direct effect). Figure 2 shows this decomposition for $\delta = 5$ in DGP 1 and DGP 2. It is readily apparent that the LP estimator closely tracks the direct effect, which would be sufficient if H_t were exogenously determined with respect to the macroeconomy, but does not fully capture the indirect effect of perturbing ε_{1t} by δ on the model parameters that arises when H_t is endogenous, as suggested by the analysis in Section 3.2. Since larger positive δ values make it more likely that a policy shock would catapult the economy from recession to expansion, conditional on being in a recession, the LP estimator becomes increasingly inaccurate. Additional simulation results in online Appendix D show that the magnitude of this indirect effect, and hence the

[7] The state-dependent LP estimator implicitly sets the shock size δ to unity, which in general differs from one standard deviation of the structural error. However, given that δ in our data generating process is a multiple of the standard deviation of ε_{1t}, responses to δ standard deviations may be constructed by scaling the estimated LP response function by a factor of δ.

asymptotic bias, increases with $\gamma_E - \gamma_R$ and could be even larger than in the examples shown here.

5 Revisiting the government spending multiplier

What matters from an applied point of view is not only that the state-dependent LP estimator is inconsistent when H_t is endogenous and δ is non-negligible, as we have shown, or how large the asymptotic bias of the LP response estimator is, but also how large the bias of the corresponding cumulative multipliers is. In this section, we quantify this bias for $\delta \in \{1, 5, 10\}$, given a prototypical data generating process motivated by Ramey and Zubairy's (2018) widely cited study of the government spending multiplier in good times and bad times. Given that Ramey and Zubairy employed state dependent local projections, they did not specify the underlying DGP, but the following process is one empirically plausible representation of such as DGP.

Let $z_t = (x_t, g_t, y_t)'$ where x_t denotes Ramey and Zubairy's military spending news measure in period t relative to potential GDP in period $t-1$, g_t is real government spending in period t relative to potential GDP in period $t-1$, and y_t is real GDP in period t relative to potential GDP in period $t-1$. The data are quarterly. Consider a trivariate state-dependent data generating process given by

$$C_{t-1} \begin{bmatrix} x_t \\ g_t \\ y_t \end{bmatrix} = k_{t-1} + B_{t-1}(L) \begin{bmatrix} x_{t-1} \\ g_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} \tag{10}$$

where

$$\begin{aligned} k_{t-1} &= H_{t-1} k_E + (1 - H_{t-1})\, k_R, \\ C_{t-1} &= H_{t-1} C_E + (1 - H_{t-1})\, C_R, \\ B_{t-1}(L) &= H_{t-1} B_E(L) + (1 - H_{t-1})\, B_R(L), \end{aligned} \tag{11}$$

ε_t is a vector of mutually independent $N(0,1)$ population innovations, $B_E(L)$ and $B_R(L)$ are lagged polynomials, and the coefficient matrices are of suitable dimensions. Thus, $\delta = 1$ corresponds to a one standard-deviation shock. The model allows for four autoregressive lags. We obtain the parameter values by fitting the model to the historical data used by Ramey and Zubairy (2018). To conserve space, the parameter values used in the simulation study are reported in online Appendix C.

Ramey and Zubairy (2018) considered several alternative definitions of economic expansions including periods when the unemployment rate is below the sample mean, positive deviations from the Hodrick-Prescott trend of the unemployment rate, and expansion dates determined by the NBER business cycle committee.[8] Here we consider two expansion indicators for expository purposes. In one DGP, expansions are defined as periods when output exceeds potential output in period $t-1$,

$$H_t = \begin{cases} 1 \text{ if } y_t \ > 1 \\ 0 \quad \text{otherwise,} \end{cases}$$

whereas the other DGP defines expansions as periods where real GDP in t, relative to potential GDP in period $t-1$, exceeds a twelve-quarter trailing average,

$$H_t = \begin{cases} 1 \text{ if } y_t > \frac{1}{12}\sum_{i=1}^{12} y_{t-i} \\ \quad 0 \quad \text{otherwise.} \end{cases} \qquad .$$

These definitions are similar to those used in Alloza (2022), for example. We further assume that x_t follows an $AR(4)$ process to account for serial correlation in Ramey and Zubairy's military spending news measure (see Alloza, Gonzalo and Sanz (2021).

Consider the dynamic effects of a shock of magnitude δ in ε_{1t} on g_{t+h} and y_{t+h}. Recall that the population impulse response is defined as $CAR_{g,i}\left(\delta,\overline{h}\right) = E\left(g_{t+i}\left(\varepsilon_{1t}+\delta\right) - g_{t+i}\left(\varepsilon_{1t}\right)|H_{t-1} = \overline{h}\right)$ and $CAR_{y,i}\left(\delta,\overline{h}\right) = E\left(y_{t+i}\left(\varepsilon_{1t}+\delta\right) - y_{t+i}\left(\varepsilon_{1t}\right)|H_{t-1} = \overline{h}\right)$, whereas the LP estimands $b_{g,i}\left(\overline{h}\right)$ and $b_{y,i}\left(\overline{h}\right)$ are defined as previously. In this section, we follow Ramey and Zubairy (2018) in focusing on the implied fiscal multipliers, defined as the relative response of output and government spending to a military spending news shock. The cumulative fiscal multiplier over the h-quarter horizon is defined as

$$\mathcal{M}_h\left(\delta,\overline{h}\right) = \frac{\sum_{i=0}^{h} CAR_{y,i}\left(\delta,\overline{h}\right)}{\sum_{i=0}^{h} CAR_{g,i}\left(\delta,\overline{h}\right)},$$

in population, whereas the corresponding fiscal multiplier based on the LP estimates is

$$M_h\left(\overline{h}\right) = \frac{\sum_{i=0}^{h} b_{y,i}\left(\overline{h}\right)}{\sum_{i=0}^{h} b_{g,i}\left(\overline{h}\right)}.$$

The number of draws used to compute these conditional expectations is 120 million to ensure that the multiplier on impact matches the population multiplier.

[8]Since the NBER business cycle dates are based on data that are correlated with the endogenous model variables, this approach does not avoid the concerns discussed in Section 3.

Figure 3 contains the results for H_{t-1} depending on the deviation from potential output in the previous period, while Figure 4 contains the results for H_{t-1} defined as a function of the one-sided $MA(12)$ filter. The plots illustrate that, at horizon $h = 0$, the LP estimator of the fiscal multiplier recovers the population response for all values of δ, even when H_t is endogenous, consistent with Proposition 4. However, for larger h, the LP estimator diverges from the population multiplier, especially when the economy is in recession. There is clear evidence of large asymptotic bias (expressed in percentage deviations from the population cumulative multiplier) that is increasing in δ, reaching 45 percent in some cases. In general, the extent of the asymptotic bias depends on the parameter values, the functional form of H_t, the magnitude and sign of the shock δ, and the state of the economy at the time of the policy shock.

Applied researchers most commonly report the two-year and four-year cumulative fiscal multiplier. Table 2 summarizes, for each DGP, the asymptotic bias of these cumulative multipliers for various δ. Whereas for $\delta = 1$ the asymptotic bias of the LP estimator is at most 4% in absolute terms, the bias tends to increases with the magnitude of δ, especially when the economy is in a recession at the time of the government spending shock. When $H_t = 1(y_t > 1)$, given a 5 standard deviation shock, the asymptotic bias of the LP estimator is 18% for the four-year cumulative multiplier, conditional on being in a recession, and 7% conditional on the economy being in expansion. Broadly similar results are obtained for $H_t = 1(y_t > MA(12))$. For example, the bias of the LP estimator of the four year integral is as large as 11% conditional on a recession and as large as 8% conditional on an expansion. For the two-year cumulative multiplier the biases are more modest. Given a 10 standard deviation shock, the asymptotic bias of the state-dependent LP estimator of the four-year cumulative fiscal multiplier reaches close to 40 percent, given a recession. This evidence suggests that the asymptotic bias of the LP estimator can be large enough in realistic settings to be a concern for applied work.[9] We also examined analogous results for $\delta \in \{-1, -5, -10\}$, which are reported in Appendix D, and found asymptotic biases in the cumulative fiscal multiplier that were larger than those for the corresponding positive δ conditional on expansions, and somewhat smaller, but still substantial, conditional on recessions.

[9]One might have expected that the bias in the numerator and denominator of the multiplier would offset, leaving the multiplier largely unaffected. Our simulations show this not need be the case in general. The asymptotic biases in the responses of GDP and government spending have the same sign at some horizons, but opposite signs at other horizons. Moreover, even when the bias in the LP estimator of the government spending and GDP responses are of the same sign, their magnitude differs and, hence. the bias in the cumulative responses typically does not cancel.

It may be tempting to argue that our evidence that the asymptotic bias is at most 4 percent for $\delta = 1$ suggests that as long as δ is not too large, the state-dependent LP estimator will get the multiplier right at least approximately. That appears true in this example, but we cannot rule out that for other specifications of the data generating process this bias could be larger. As we discuss in the next section, a better option would be to replace the state-dependent LP estimator by an alternative estimator that remains asymptotically valid in this setting.

6 An alternative to state-dependent local projections

As we demonstrated, state-dependent local projections do not in general recover the conditional average response function of y_{t+h} to a shock in ε_{1t} of fixed size δ, yet macroeconomists often care about the impact of large shocks. This poses a quandary for applied researchers. In this section, we propose an alternative approach that can be used when the conditional average response function is the impulse response function of interest. While fully developing the theoretical foundations of this estimator (or examining its finite-sample accuracy) is beyond the scope of this paper, we outline the central idea underlying this proposal. Let

$$g_h\left(e,\bar{h}\right) \equiv E\left(y_{t+h}|\varepsilon_{1t} = e, H_{t-1} = \bar{h}\right),$$

define the expectation of y_{t+h}, conditional on $\varepsilon_{1t} = e$ and $H_{t-1} = \bar{h}$ for any fixed values e and $\bar{h}$ in the support of ε_{1t} and H_{t-1}, respectively.

Then, under our model assumptions (i.e., model (3) and (4) and Assumptions 1 and 2), we can write $CAR_h\left(\delta,\bar{h}\right)$ as $E\left(g_h\left(\varepsilon_{1t}+\delta,\bar{h}\right) - g_h\left(\varepsilon_{1t},\bar{h}\right)\right)$. To see this, note that by the law of iterated expectations,

$$\begin{aligned}
& E\left[y_{t+h}\left(e+\delta\right) - y_{t+h}\left(e\right)|H_{t-1} = \bar{h}\right] \\
= \; & E\left[y_{t+h}\left(e+\delta\right)|\varepsilon_{1t} = e+\delta, H_{t-1} = \bar{h}\right] - E\left[y_{t+h}\left(e\right)|\varepsilon_{1t} = e, H_{t-1} = \bar{h}\right] \\
= \; & E\left[y_{t+h}|\varepsilon_{1t} = e+\delta, H_{t-1} = \bar{h}\right] - E\left[y_{t+h}|\varepsilon_{1t} = e, H_{t-1} = \bar{h}\right] \\
\equiv \; & g_h\left(e+\delta,\bar{h}\right) - g_h\left(e,\bar{h}\right),
\end{aligned}$$

where the second equality follows by the independence between the potential outcomes $y_{t+h}\left(e\right)$ and ε_{1t} (see Lemma A.1 in online Appendix A), and the third equality follows because $y_{t+h}\left(e\right) =$

y_{t+h} when $\varepsilon_{1t} = e$ and $y_{t+h}(e+\delta) = y_{t+h}$ when $\varepsilon_{1t} = e+\delta$. It follows that

$$\begin{aligned} CAR_h\left(\delta, \bar{h}\right) &\equiv E\left(y_{t+h}\left(\varepsilon_{1t}+\delta\right) - y_{t+h}\left(\varepsilon_{1t}\right)|H_{t-1} = \bar{h}\right) \\ &= E\left(g_h\left(\varepsilon_{1t}+\delta, \bar{h}\right) - g_h\left(\varepsilon_{1t}, \bar{h}\right)|H_{t-1} = \bar{h}\right) \\ &= E\left(g_h\left(\varepsilon_{1t}+\delta, \bar{h}\right) - g_h\left(\varepsilon_{1t}, \bar{h}\right)\right), \end{aligned}$$

where the last equality follows because ε_{1t} is independent of $H_{t-1} = \eta\left(w_s : s \leq t-1\right)$ under Assumptions 1 and 2.

This result suggests the following approach to estimating $CAR_h\left(\delta, \bar{h}\right)$. First, we estimate $g_h\left(e, \bar{h}\right) = E\left(y_{t+h}|\varepsilon_{1t} = e, H_{t-1} = \bar{h}\right)$ consistently using the observed sample $\{y_{t+h}, \varepsilon_{1t}, H_{t-1}\}$. When $x_t = \varepsilon_{1t}$, as in the narrative approach to identification, $g_h\left(e, \bar{h}\right)$ is identified from data for these three variables. Since this function is generally a complicated nonlinear function of e and $\bar{h}$ when H_t is endogenous, we can use a nonparametric regression of y_{t+h} on ε_{1t} and H_{t-1} in this step. Letting $\hat{g}_h\left(e, \bar{h}\right)$ denote this estimator, we then average the difference $\hat{g}_h\left(\varepsilon_{1t}+\delta, \bar{h}\right) - \hat{g}_h\left(\varepsilon_{1t}, \bar{h}\right)$ over the realizations of ε_{1t} in the sample.

Algorithm 6.1 (Nonparametric CAR) *Given a sample $\{y_t, \varepsilon_{1t}, q, : t = 1, \ldots, T\}$, proceed in two steps:*

1. *Obtain the nonparametric estimator $\hat{g}_h\left(e, \bar{h}\right) \equiv \widehat{E}\left(y_{t+h}|\varepsilon_{1t} = e, H_{t-1} = \bar{h}\right)$ of $g_h\left(e, \bar{h}\right)$.*
2. *Estimate $CAR_h\left(\delta, \bar{h}\right)$ as*

$$\widehat{CAR}_h\left(\delta, \bar{h}\right) = \frac{1}{T}\sum_{t=1}^{T}\left(\hat{g}_h\left(\varepsilon_{1t}+\delta\right) - \hat{g}_h\left(\varepsilon_{1t}\right)\right).$$

This proposal provides a constructive alternative to the use of state-dependent LP estimators in applied work, when δ is not negligible and the state of the economy is endogenous. How well this new nonparametric estimator of the population response works in practice is the subject of ongoing research.

7 Conclusion

When the state of the economy evolves independently of macroeconomic shocks, state-dependent local projections recover in the limit the conditional response function to a shock of size δ, whether δ is fixed or infinitesimal. More typically, the state of the economy is defined in terms of endogenous model variables such as real output or unemployment. A case in point are recessions as defined by the NBER business cycle committee or the rule of thumb defining recessions as two consecutive quarters of negative output growth. Other typical examples include recessions defined as negative deviations of real output from a moving-average trend or Hodrick-Prescott filter (HP) trend or recessions defined as unemployment rates exceeding some threshold.

When the state of the economy is endogenous with respect to macroeconomic shocks, state-dependent local projections under suitable assumptions will recover the conditional marginal response function with respect to an infinitesimal structural shock. However, they will not recover the response of the economy to larger structural shocks, conditional on being in a recession or an expansion at the time of this shock. As a result, state-dependent local projections should not be used to quantify the importance of large historical fiscal policy shocks, for example. Nor do they seem well suited for analyzing the impact of major fiscal policy interventions during the Great Recession or the Covid-19 Recession. Analogous statements apply to studies of monetary policy shocks or uncertainty shocks based on state-dependent local projections.

These problems may in principle be overcome by the use of state-dependent structural VAR models at the cost of added complexity. We noted that our analysis suggests an alternative nonparametric approach that retains the parsimony and ease of estimation of LP estimators, yet preserves the ability to recover the average response of the outcome variable, conditional on the state of the economy at the time of the shock. How well this new nonparametric estimator of the population response works in practice is the subject of ongoing research.

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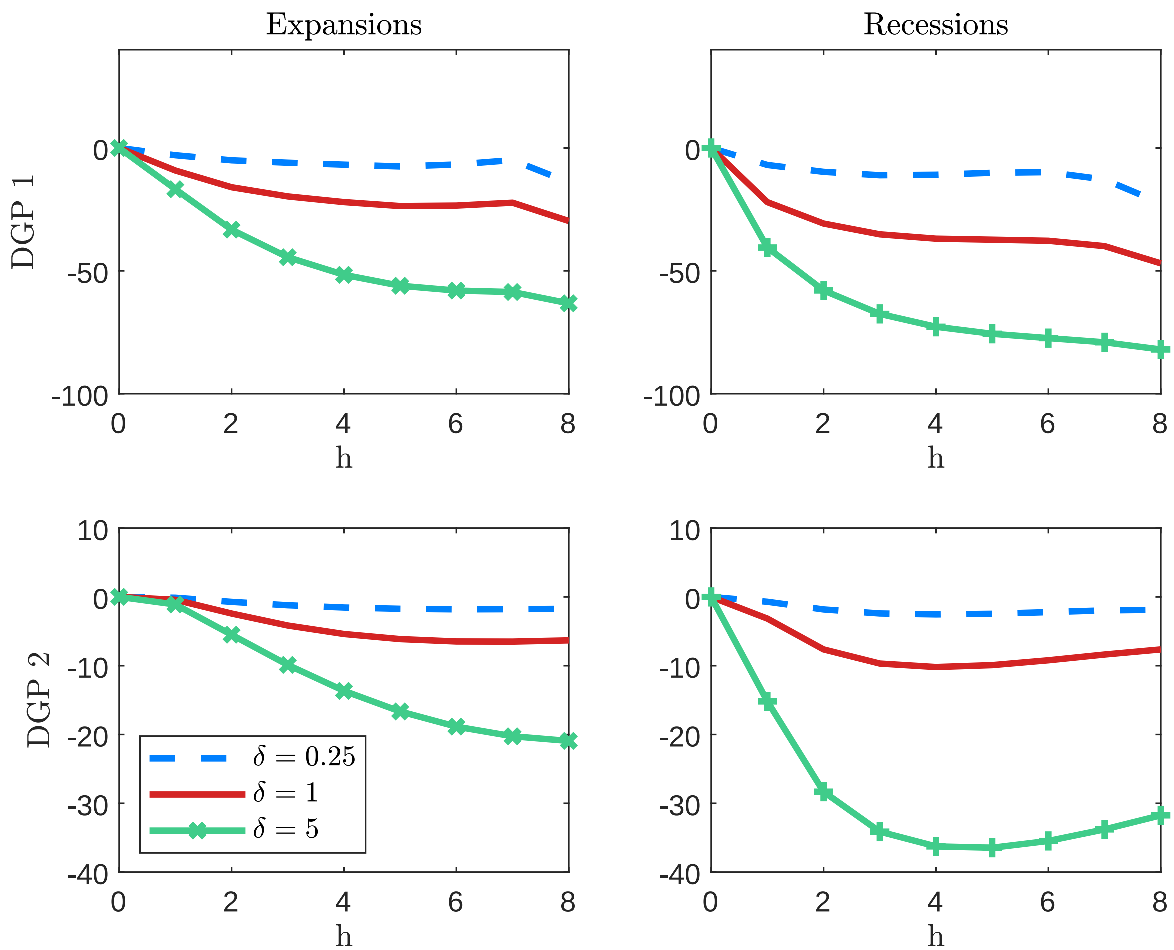

Figure 1: Asymptotic bias of LP response when $H_t = 1\,(y_t > 0)$

Table 1: Selected journal articles that employ state-dependent local projections

Fiscal policy	**Fiscal policy (continued)**
Alesina et al., 2018	Riera-Crichton, Vegh, Vuletin, 2015
Alesina, Favero, Giavazzi, 2019	Sheremirov, Spirovska, 2022
Alloza, 2022	
Auerbach, Gorodnichenko, 2013	**Monetary policy**
Auerbach, Gorodnichenko, 2016	Albrizio et al., 2020
Bachmann, Sims, 2012	Albuquerque, 2019
Berge, De Ridder, Pfajfar, 2021	Alpanda, Granziera, Zubairy, 2021
Bernardini, Peersman, 2018	Angrist, Jordà, Kuersteiner, 2018
Bernardini, De Schryder, Peersman, 2020	Ascari, Haber, 2022
Biolsi, 2017	Auer, Bernardini, Cecioni, 2021
Boehm, 2020	Barnichon, Matthes, 2018
Born, Müller, Pfeifer, 2020	El Herradi, Leroy, 2021
Caggiano et al., 2015	Falck, Hoffmann, Hürtgen, 2021
Candelon, Lieb, 2013	Furceri, Loungani, Zdzienicka, 2018
Choi, Shin, Yoo, 2022	Jorda, Schularick, Taylor, 2020
Demirel, 2021	Santoro et al., 2014
El-Shagi, von Schweinitz, 2021	Tenreyro, Thwaites, 2016
Eminidou, Geiger, Zachariadis, 2023	
Ghassibe, Zanetti, 2022	**Uncertainty**
Jorda, Taylor, 2016	Cacciatore, Ravenna, 2021
Klein, 2017	Caggiano, Castelnuovo, Groshenny, 2014
Klein, Polattimur, Winkler, 2022	Tillmann, 2020
Klein, Winkler, 2021	
Leduc, Wilson, 2012	**Other**
Liu, 2022	De Haan, Wiese, 2022
Liu, 2023	Duval, Furceri, 2018
Miyamoto, Nguyen, Segeyev, 2018	Lastauskas, Stakenas, 2020
Miyamoto, Nguyen, Sheremirov, 2019	Loipersberger, Matschke, 2022
Owyang, Ramey, Zubairy, 2013	Sheng, Sukaj, 2021
Ramey, Zubairy, 2018	

Note: The articles listed above appeared in: American Economic Review, American Economic Journal: Macroeconomics, Economic Journal, European Economic Review, IMF Economic Review, International Economic Review, International Journal of Central Banking, Journal of Applied Econometrics, Journal of Economic Dynamics and Control, Journal of Economic Perspectives, Journal of International Economics, Journal of International Money and Finance, Journal of Monetary Economics, Journal of Money, Credit and Banking, Journal of Political Economy, Journal of Public Economics, NBER Macroeconomics Annual, Review of Economics and Statistics.

Table 2: Asymptotic bias in the cumulative multiplier

	DGP1		DGP2	
	Expansion	Recession	Expansion	Recession
$\delta = 1$				
Bias in 2 year integral	1.5	1.6	0.3	0.7
Bias in 4 year integral	2.3	3.7	1.1	2.2
$\delta = 5$				
Bias in 2 year integral	2.5	6.9	4.3	3.3
Bias in 4 year integral	6.6	17.9	7.7	11.1
$\delta = 10$				
Bias in 2 year integral	1.7	13.0	5.5	5.7
Bias in 4 year integral	10.2	36.6	12.5	21.1

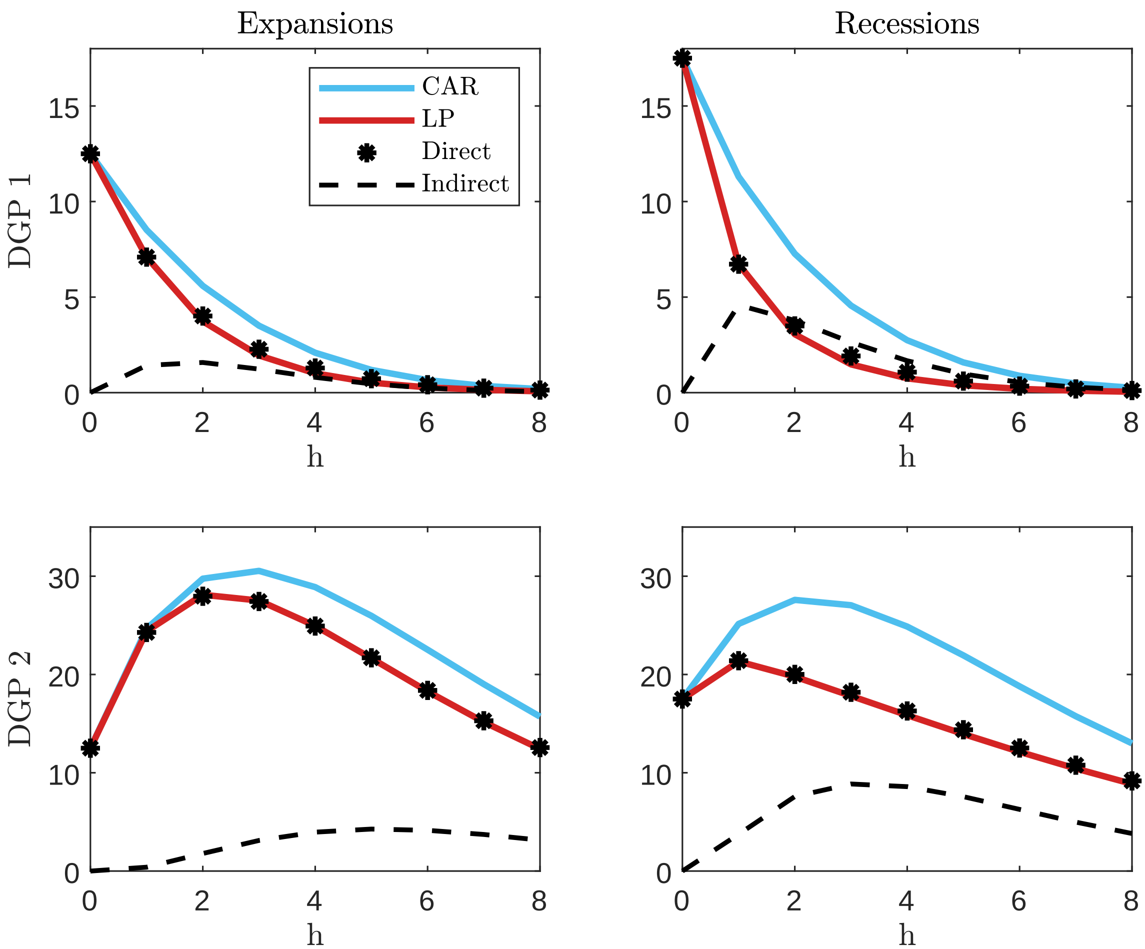

Figure 2: LP response and decomposition of CAR when $H_t = 1\,(y_t > 0)$ and $\delta = 5$

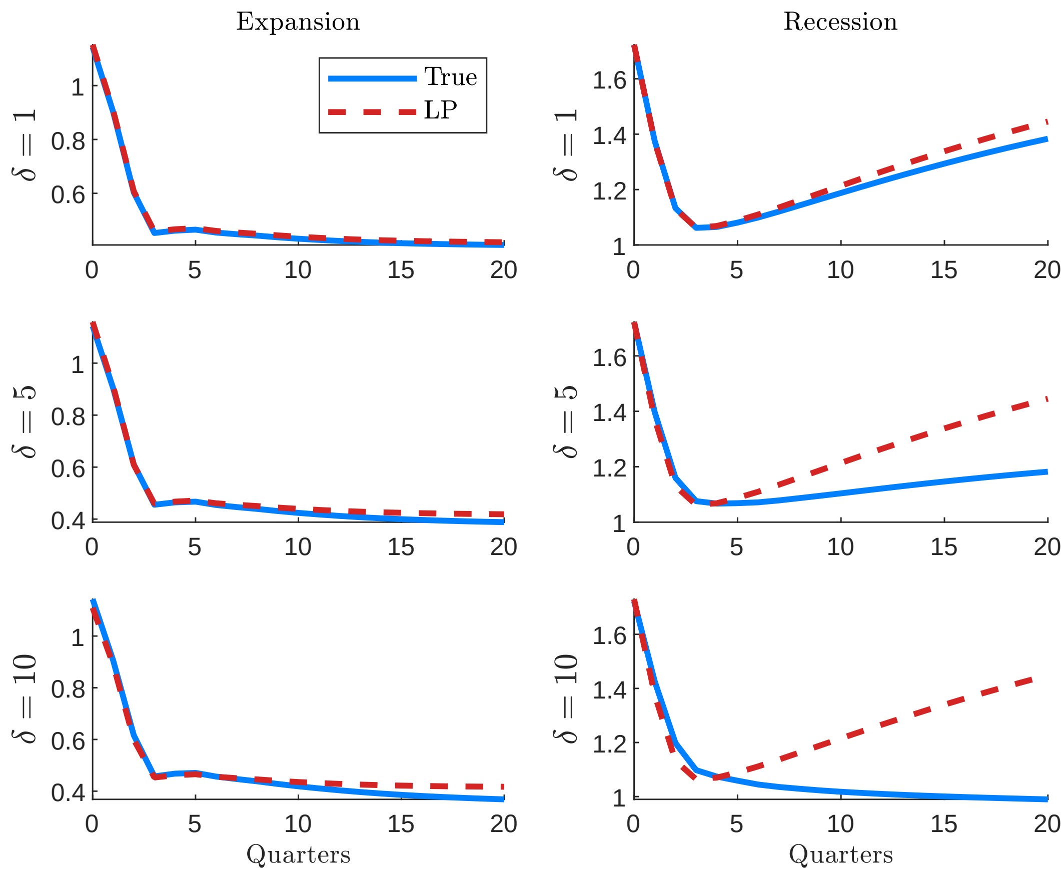

Figure 3: Cumulative spending multiplier when $H_t = 1\,(y_t > 1)$

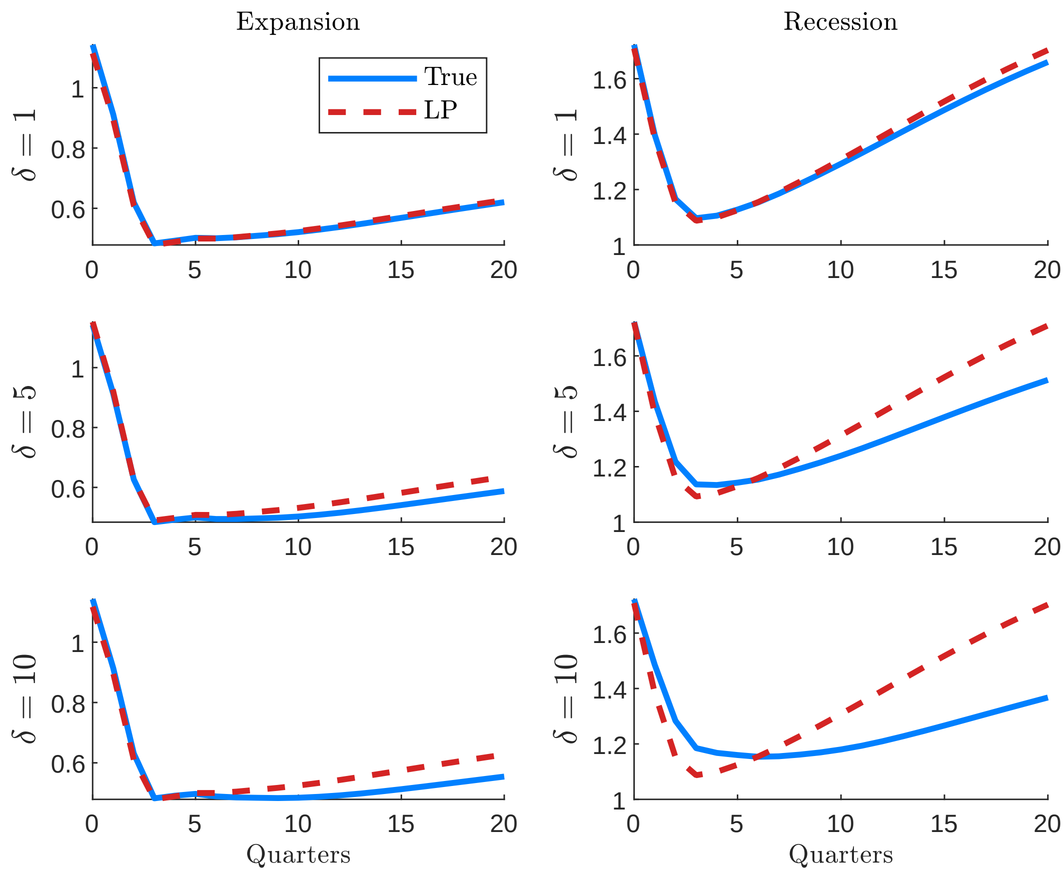

Figure 4: Cumulative spending multiplier when $H_t = 1\,(y_t > MA(12))$

For online publication:
Appendix to "State-dependent local projections"*

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April 11, 2023

This online appendix consists of four appendices. Appendix A contains the proofs of the main propositions in the paper. Appendix B provides additional theoretical results for a multivariate state-dependent structural VAR model when H_t is exogenous. These results generalize Propositions 3.1 and 3.2 in the main text to a multivariate setting where ε_{1t} is identified within the structural VAR model. Appendix C describes the parameter values used in the data generating process of Section 5. Finally, Appendix D contains additional simulation results.

*Acknowledgments: The views expressed in this paper are our own and should not be interpreted as reflecting the views of the Federal Reserve Bank of Dallas or any other member of the Federal Reserve System. An earlier version of this paper circulated under the title "When do state-dependent local projections work?". We thank Mikkel Plagborg-Møller for helpful comments.

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A Proofs of the main propositions

The proof of our results relies on the independence between the potential outcomes $y_{t+h}(e)$ and the structural error ε_{1t}. This independence condition follows straightforwardly from our assumptions and is instrumental in providing a causal interpretation to the state-dependent LP estimands. We summarize this result in the following lemma.

Lemma A.1 *Consider the structural process defined by equations (3) and (4) in the main text. Under Assumptions 1 and 2, ε_{1t} is independent of $\{y_{t+h}(e),\ e \in A\}$, where A is the support of ε_{1t}.*

Proof of Lemma A.1. This proof is obvious given the definitions of $y_{t+h}(e)$ derived in the main text. ■

Proof of Lemma 2.1. Let $y_{t+h}(e) = m_h(e, U_{t+h})$. For given e, we can write

$$E\left(y_{t+h}(e+\delta) - y_{t+h}(e)\,|H_{t-1} = \bar{h}\right) = \int \left[m_h(e+\delta, U) - m_h(e, U)\right] f\left(U|\bar{h}\right) dU,$$

where $f\left(U|\bar{h}\right)$ denotes the conditional density function of U_{t+h} given $H_{t-1} = \bar{h}$. Dividing by δ and integrating with respect to e yields

$$\int_A \delta^{-1} E\left(y_{t+h}(e+\delta) - y_{t+h}(e)\,|H_{t-1} = \bar{h}\right) f\left(e|\bar{h}\right) de = \int_A \int \delta^{-1}\left[m_h(e+\delta, U) - m_h(e, U)\right] f\left(U|\bar{h}\right) f\left(e|\bar{h}\right) dU,$$

where $f\left(e|\bar{h}\right)$ denotes the conditional density function of ε_{1t} given $H_{t-1} = \bar{h}$. Under the assumption that ε_{1t} and U_{t+h} are independent, conditionally on $H_{t-1} = \bar{h}$, we have that $f\left(e, U|\bar{h}\right) = f\left(U|\bar{h}\right) f\left(e|\bar{h}\right)$. Moreover, for fixed e and U, by the definition of a derivative, $\lim_{\delta\to 0} \delta^{-1}\left[m_h(e+\delta, U) - m_h(e, U)\right] = m_h'(e, U)$, assuming the derivative of m_h with respect to e exists. Thus,

$$\begin{aligned}
& \lim_{\delta\to 0} \delta^{-1} CAR_h\left(\delta, \bar{h}\right) \\
= & \lim_{\delta\to 0} \int_A \delta^{-1} E\left(y_{t+h}(e+\delta) - y_{t+h}(e)\,|H_{t-1} = \bar{h}\right) f\left(e|H_{t-1} = \bar{h}\right) de \\
= & \int_A \int_U m_h'(e, U)\, f\left(e, U|\bar{h}\right) dedU \\
= & E\left(m_h'(\varepsilon_{1t}, U_{t+h})\,|H_{t-1} = \bar{h}\right) = E\left(y_{t+h}'(\varepsilon_{1t})\,|H_{t-1} = \bar{h}\right) \equiv CMR_h\left(\bar{h}\right),
\end{aligned}$$

where the last equality follows by definition of $y_{t+h} = m_h(\varepsilon_{1t}, U_{t+h})$. ■

Proof of Proposition 3.1. The proof is in the text. ■

Proof of Proposition 3.2. The proof is in the text. ■

Proof of Proposition 3.3. We start by deriving the potential outcomes $y_{t+h}(e)$ for this model.

For any e, define

$$\beta(e) = \beta_R + (\beta_E - \beta_R)\,\eta(e) \text{ and } \gamma(e) = \gamma_R + (\gamma_E - \gamma_R)\,\eta(e),$$

with $\eta(e) = 1(e > c)$ for any fixed e. Let $V_{0t} \equiv \gamma_{t-1}y_{t-1} + \varepsilon_{2t}$ be a function of $(\varepsilon_{2t}, y_{t-1}, \varepsilon_{1t-1}) = (\varepsilon_{2t}, z'_{t-1}) \equiv U'_t$, since $x_t = \varepsilon_{1t}$ and $z'_t = (x_t, y_t)$. With this notation, for $h = 0$, $y_t = \beta_{t-1}\varepsilon_{1t} + V_{0t}$. The potential outcome for $h = 0$ is obtained from this equation by fixing $\varepsilon_{1t} = e$:

$$y_t(e) = \beta_{t-1}e + V_{0t} \equiv m_0(e, U_t),$$

with $U_t \equiv (\varepsilon_{2t}, z'_{t-1})'$. For $h = 1$, $y_{t+1} = \beta_t\varepsilon_{1t+1} + \gamma_t y_t + \varepsilon_{2t+1}$, where $y_t = y_t(\varepsilon_{1t})$, $\beta_t = \beta(\varepsilon_{1t})$ and $\gamma_t = \gamma(\varepsilon_{1t})$. Hence, upon fixing $\varepsilon_{1t} = e$, we have that

$$y_{t+1}(e) = \beta(e)\,\varepsilon_{1t+1} + \gamma(e)\,y_t(e) + \varepsilon_{2t+1},$$

which shows that $y_{t+1}(e)$ can be obtained from $y_t(e)$. Replacing $y_t(e) = \beta_{t-1}e + V_{0t}$,

$$y_{t+1}(e) = \gamma(e)\,\beta_{t-1}e + V_{t+1}(e) \equiv m_1(e, U_{t+1}), \tag{1}$$

where

$$V_{t+1}(e) = \gamma(e)\,V_{0t} + \beta(e)\,\varepsilon_{1t+1} + \varepsilon_{2t+1} \equiv V_1(e, U_{t+1})$$

with

$$U_{t+1} = (\varepsilon'_{t+1}, \varepsilon_{2t}, z'_{t-1})' \equiv (\varepsilon'_{t+1}, U'_t).$$

For $h = 2$, writing $\beta_{t+1} \equiv \beta(\varepsilon_{1t+1})$ and $\gamma_{t+1} \equiv \gamma(\varepsilon_{1t+1})$, it follows that

$$\begin{aligned} y_{t+2}(e) &= \beta_{t+1}\varepsilon_{1t+2} + \gamma_{t+1}y_{t+1}(e) + \varepsilon_{2t+2} \\ &= \beta_{t+1}\varepsilon_{1t+2} + \gamma_{t+1}[\gamma(e)\,\beta_{t-1}e + V_{t+1}(e)] + \varepsilon_{2t+2} \\ &= \gamma_{t+1}\gamma(e)\,\beta_{t-1}e + V_{t+2}(e) \equiv m_2(e, U_{t+1}), \end{aligned}$$

where

$$\begin{aligned} V_{t+2}(e) &\equiv \gamma_{t+1}V_{t+1}(e) + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2} \\ &= \gamma_{t+1}[\gamma(e)\,V_{0t} + \beta(e)\,\varepsilon_{1t+1} + \varepsilon_{2t+1}] + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2} \\ &= \gamma_{t+1}\gamma(e)\,V_{0t} + \gamma_{t+1}\beta(e)\,\varepsilon_{1t+1} + \varepsilon_{2t+1} + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2}, \end{aligned}$$

which is a function of $U_{t+2} \equiv (\varepsilon'_{t+2}, \varepsilon'_{t+1}, \varepsilon_{2t}, z'_{t-1})' = (\varepsilon'_{t+2}, U'_{t+1})'$. For any $h > 1$,

$$y_{t+h}(e) = \gamma_{t+h-1}\cdots\gamma_{t+1}\gamma(e)\,\beta_{t-1}e + V_{t+h}(e) \equiv m_h(e, U_{t+h}),$$

where

$$V_{t+h}(e) \equiv \gamma_{t+h-1} V_{t+h-1}(e) + \beta_{t+h-1}\varepsilon_{1t+h} + \varepsilon_{2t+h},$$

and $U_{t+h} \equiv \left(\varepsilon'_{t+h}, U'_{t+h-1}\right)'$.

Next, we show part (i) of the proposition, which derives the conditional average response function for any fixed δ. For $h = 0$, $y_t(e+\delta) - y_t(e) = \beta_{t-1}\delta$, which does not depend on e. Hence,

$$CAR_0\left(\delta, \bar{h}\right) = E\left(y_t(\varepsilon_{1t}+\delta) - y_t(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right) = E\left(\beta_{t-1} | H_{t-1} = \bar{h}\right)\delta = \beta_{\bar{h}}\delta.$$

For $h = 1$, by Definition 1,

$$CAR_1\left(\delta, \bar{h}\right) = E\left(y_{t+1}(\varepsilon_{1t}+\delta) - y_{t+1}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right),$$

where $y_{t+1}(\varepsilon_{1t})$ is equal to $y_{t+1}(e)$ with $e = \varepsilon_{1t}$ (and similarly for $y_{t+1}(\varepsilon_{1t}+\delta)$). We will evaluate $CAR_1\left(\delta, \bar{h}\right)$ below, but note that under the simplified Assumption 3, for any $h > 1$, we can write $CAR_h\left(\delta, \bar{h}\right)$ as a function of $CAR_1\left(\delta, \bar{h}\right)$. Specifically, for $h = 2$, we have that

$$\begin{aligned} y_{t+2}(e+\delta) - y_{t+2}(e) &= \gamma_{t+1} y_{t+1}(e+\delta) + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2} - \left(\gamma_{t+1} y_{t+1}(e) + \beta_{t+1}\varepsilon_{1t+2} + \varepsilon_{2t+2}\right) \\ &= \gamma_{t+1}\left[y_{t+1}(e+\delta) - y_{t+1}(e)\right], \end{aligned}$$

and more generally for any $h > 1$,

$$y_{t+h}(e+\delta) - y_{t+h}(e) = \gamma_{t+h-1}\left[y_{t+h-1}(e+\delta) - y_{t+h-1}(e)\right] = \left(\gamma_{t+h-1}\cdots\gamma_{t+1}\right)\left[y_{t+1}(e+\delta) - y_{t+1}(e)\right].$$

By Definition 1, for any $h > 1$,

$$\begin{aligned} CAR_h\left(\delta, \bar{h}\right) &= E\left[y_{t+h}(\varepsilon_{1t}+\delta) - y_{t+h}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right] \\ &= E\left(\gamma_{t+h-1}\cdots\gamma_{t+1}\right) E\left[y_{t+1}(\varepsilon_{1t}+\delta) - y_{t+1}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right] \\ &= (\bar{\gamma})^{h-1} CAR_1\left(\delta, \bar{h}\right), \end{aligned} \tag{2}$$

where we let $\bar{\gamma} \equiv E(\gamma_{t+1})$ for any t. The last equality follows from the fact that γ_t is a function of ε_{1t} and ε_{1t} is i.i.d. This implies that we only need to evaluate $CAR_1\left(\delta, \bar{h}\right)$ and $\bar{\gamma}$ to obtain the entire conditional average response function. Under Assumption 3(a) and (b), where the Gaussianity assumption is instrumental in deriving the closed form expressions for $\bar{\gamma}$ and $CAR_1\left(\delta, \bar{h}\right)$, using (1), for any fixed e,

$$\begin{aligned} y_{t+1}(e+\delta)-y_{t+1}(e) &= \gamma(e)\beta_{t-1}\delta \\ &+[\gamma(e+\delta)-\gamma(e)]\beta_{t-1}\delta \\ &+[\gamma(e+\delta)-\gamma(e)]\beta_{t-1}e \\ &+[\gamma(e+\delta)-\gamma(e)]V_{0t} \\ &+[\beta(e+\delta)-\beta(e)]\varepsilon_{1t+1}. \end{aligned}$$

Next, evaluate this difference at $e=\varepsilon_{1t}$ and take the expectation, conditionally on $H_{t-1}=\bar{h}$. It follows that for any fixed δ,

$$\begin{aligned} CAR_1\left(\delta,\bar{h}\right) &\equiv E\left[y_{t+1}(\varepsilon_{1t}+\delta)-y_{t+1}(\varepsilon_{1t})|H_{t-1}=\bar{h}\right] \\ &= E\left[\gamma(\varepsilon_{1t})|H_{t-1}=\bar{h}\right]\beta_{\bar{h}}\delta+\{E\left[(\gamma(\varepsilon_{1t}+\delta)-\gamma(\varepsilon_{1t}))|H_{t-1}=\bar{h}\right]\beta_{\bar{h}}\delta \\ &\quad +E\left[(\gamma(\varepsilon_{1t}+\delta)-\gamma(\varepsilon_{1t}))\varepsilon_{1t}|H_{t-1}=\bar{h}\right]\beta_{\bar{h}}+E\left[(\gamma(\varepsilon_{1t}+\delta)-\gamma(\varepsilon_{1t}))V_{0t}|H_{t-1}=\bar{h}\right] \\ &\quad +E[(\beta(\varepsilon_{1t}+\delta)-\beta(\varepsilon_{1t}))\varepsilon_{1t+1}|H_{t-1}=\bar{h}]\} \end{aligned} \tag{3}$$

Note that the last term in (3) has conditional mean zero. This follows by the law of iterated expectations, using the fact that ε_{1t} is an i.i.d. zero mean random variable which is independent of ε_{2t}. Under these assumptions, V_{0t} is independent of ε_{1t}, and the second-to-last term can be written as $E(\gamma(\varepsilon_{1t}+\delta)-\gamma(\varepsilon_{1t}))]v_{\bar{h}}$ (where $v_{\bar{h}}=E\left(V_{0t}|H_{t-1}=\bar{h}\right)=\gamma_{\bar{h}}E(y_{t-1}|H_{t-1}=\bar{h})$). By using similar arguments, we can decompose $CAR_1\left(\delta,\bar{h}\right)$ into the sum of

$$\begin{aligned} \text{Direct effect} &= E(\gamma(\varepsilon_{1t}))\beta_{\bar{h}}\delta. \\ \text{Indirect effect} &= E\left[(\gamma(\varepsilon_{1t}+\delta)-\gamma(\varepsilon_{1t}))\right]\beta_{\bar{h}}\delta \\ &\quad +E\left[(\gamma(\varepsilon_{1t}+\delta)-\gamma(\varepsilon_{1t}))\varepsilon_{1t}\right]\beta_{\bar{h}} \\ &\quad +E\left[\gamma(\varepsilon_{1t}+\delta)-\gamma(\varepsilon_{1t})\right]v_{\bar{h}}. \end{aligned}$$

This decomposition shows that the first component of $CAR_1\left(\delta,\bar{h}\right)$ captures the direct effect of a shock of size δ in ε_{1t} on y_{t+h}. Since $\gamma(\varepsilon_{1t})=\gamma_t$, this is the effect of a change in ε_{1t} on y_{t+h} that keeps γ_t constant, as when H_t is exogenous. However, in the current model, $H_t=\eta(\varepsilon_{1t})$, which means that when we perturb ε_{1t} by δ, this also impacts the model parameters at time t. The last three terms in $CAR_1\left(\delta,\bar{h}\right)$ capture this "indirect effect" since they depend on the wedge between $\gamma(\varepsilon_{1t}+\delta)$ and $\gamma(\varepsilon_{1t})$.

Suppose now that $\varepsilon_{1t} \sim N\left(0, \sigma_1^2\right)$, as in Assumption 3(b). Then,

$$E\left(\eta\left(\varepsilon_{1t}+\delta\right)\right) = E\left(1\left(\varepsilon_{1t}+\delta>c\right)\right) = P\left(\varepsilon_{1t}/\sigma_1 > \left(c-\delta\right)/\sigma_1\right) = 1-\Phi\left(\left(c-\delta\right)/\sigma_1\right) = \Phi\left(-c/\sigma_1+\delta/\sigma_1\right).$$

and

$$E\left(\gamma\left(\varepsilon_{1t}+\delta\right)\right) = \gamma_R + \left(\gamma_E-\gamma_R\right)\Phi\left(-c/\sigma_1+\delta/\sigma_1\right).$$

Also, we can show that

$$\begin{aligned}
E\left[\left(\gamma\left(\varepsilon_{1t}+\delta\right)-\gamma\left(\varepsilon_{1t}\right)\right)\varepsilon_{1t}\right] &= \left(\gamma_E-\gamma_R\right)E\left[\left(\eta\left(\varepsilon_{1t}+\delta\right)-\eta\left(\varepsilon_{1t}\right)\right)\varepsilon_{1t}\right] \\
&= \left(\gamma_E-\gamma_R\right)E\left[\left(1\left(\varepsilon_{1t}+\delta>c\right)-1\left(\varepsilon_{1t}>c\right)\right)\varepsilon_{1t}\right] \\
&= \left(\gamma_E-\gamma_R\right)E\left[\left(1\left(\left(c-\delta\right)/\sigma_1<\varepsilon_{1t}/\sigma_1<c/\sigma_1\right)\right)\frac{\varepsilon_{1t}}{\sigma_1}\right]\sigma_1 \\
&= \left(\gamma_E-\gamma_R\right)\sigma_1\left[\phi\left(\left(c-\delta\right)/\sigma_1\right)-\phi\left(c/\sigma_1\right)\right] \\
&= \left(\gamma_E-\gamma_R\right)\sigma_1\left[\phi\left(-c/\sigma_1+\delta/\sigma_1\right)-\phi\left(-c/\sigma_1\right)\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
CAR_1\left(\delta,\bar{h}\right) &= E\left[\gamma\left(\varepsilon_{1t}+\delta\right)\right]\beta_{\bar{h}}\delta + E\left[\left(\gamma\left(\varepsilon_{1t}+\delta\right)-\gamma\left(\varepsilon_{1t}\right)\right)\varepsilon_{1t}\right]\beta_{\bar{h}} - E\left[\gamma\left(\varepsilon_{1t}+\delta\right)-\gamma\left(\varepsilon_{1t}\right)\right]v_{\bar{h}} \\
&= \left\{\gamma_R+\left(\gamma_E-\gamma_R\right)\Phi(-c/\sigma_1+\delta/\sigma_1)\right\}\beta_{\bar{h}}\delta + \left(\gamma_E-\gamma_R\right)\sigma_1[\phi\left(-c/\sigma_1+\delta/\sigma_1\right)-\phi\left(-c/\sigma_1\right))]\beta_{\bar{h}} \\
&\quad +\{\left(\gamma_E-\gamma_R\right)[\Phi\left(-c/\sigma_1+\delta/\sigma\right)-\Phi\left(-c/\sigma_1\right)]v_{\bar{h}}\} \\
&= \underbrace{\left\{\gamma_R+\left(\gamma_E-\gamma_R\right)\Phi(-c/\sigma_1)\right\}\beta_{\bar{h}}\delta}_{=E(\gamma_t)\beta_{\bar{h}}\delta=\text{Direct effect}} \\
&\quad + \left\{\gamma_R+\left(\gamma_E-\gamma_R\right)\left[\Phi(-c/\sigma_1+\delta/\sigma_1)-\Phi(-c/\sigma_1)\right]\right\}\beta_{\bar{h}}\delta \\
&\quad + \left\{\left(\gamma_E-\gamma_R\right)\sigma_1(\phi(-c/\sigma_1+\delta/\sigma_1)-\phi(-c/\sigma_1))\right\}\beta_{\bar{h}} \qquad (4) \\
&\quad + \left\{\left(\gamma_E-\gamma_R\right)\left[\Phi\left(-c/\sigma_1+\delta/\sigma_1\right)-\Phi\left(-c/\sigma_1\right)\right]\right\}v_{\bar{h}},
\end{aligned}$$

where the last three terms define the "Indirect effect". Plugging this expression into (2) gives the formula for $CAR_h\left(\delta,\bar{h}\right)$ for any $h>1$ and any fixed δ. Note that

$$\bar{\gamma} = E\left(\gamma_t\right) = \gamma_R + \left(\gamma_E-\gamma_R\right)\Phi(-c/\sigma_1) \text{ for all } t.$$

To prove part (ii), we use the fact that

$$\begin{aligned}
CMR_h\left(\bar{h}\right) &= \lim_{\delta\to 0}[\delta^{-1}CAR_h\left(\delta,\bar{h}\right)] \\
&= \left(\bar{\gamma}\right)^{h-1}\lim_{\delta\to 0}[\delta^{-1}CAR_1\left(\delta,\bar{h}\right)] \\
&= \left(\bar{\gamma}\right)^{h-1}CMR_1\left(\bar{h}\right),
\end{aligned}$$

where $CMR_1\left(\bar{h}\right) = \lim_{\delta\to 0} CAR_1\left(\delta, \bar{h}\right)/\delta$. In particular, by dividing (4) by δ and taking the limit as $\delta \to 0$, we get

$$CMR_1\left(\bar{h}\right) = \left\{\gamma_R + \left(\gamma_E - \gamma_R\right)\Phi(-c/\sigma_1)\right\}\beta_{\bar{h}} + I_0 + I_1 + I_2,$$

where

$$\begin{aligned}
I_0 &= \lim_{\delta\to 0} \delta^{-1}\left\{\gamma_R + \left(\gamma_E - \gamma_R\right)\left[\Phi(-c/\sigma_1 + \delta/\sigma_1) - \Phi(-c/\sigma_1)\right]\right\}\beta_{\bar{h}}\delta = 0 \\
I_1 &= \lim_{\delta\to 0} \delta^{-1}\left\{\left(\gamma_E - \gamma_R\right)\sigma_1(\phi(-c/\sigma_1 + \delta/\sigma_1) - \phi(-c/\sigma_1))\right\}\beta_{\bar{h}} \\
I_2 &= \lim_{\delta\to 0}\left[\delta^{-1}\left(\gamma_E - \gamma_R\right)\left[\Phi\left(-c/\sigma_1 + \delta/\sigma_1\right) - \Phi\left(-c/\sigma_1\right)\right]\right]v_{\bar{h}}.
\end{aligned}$$

We can evaluate I_1 and I_2 by using the following two Taylor expansions of the Gaussian pdf and cdf,

$$\begin{aligned}
\phi(-c/\sigma_1 + \delta/\sigma_1) &= \phi(-c/\sigma_1) + \phi'(-c/\sigma_1)\frac{\delta}{\sigma_1} + O\left(\delta^2\right), \\
\Phi(-c/\sigma_1 + \delta/\sigma_1) &= \Phi(-c/\sigma_1) + \Phi'(-c/\sigma_1)\frac{\delta}{\sigma_1} + O\left(\delta^2\right),
\end{aligned}$$

where $\Phi'(-c/\sigma_1) = \phi(-c/\sigma_1) = \phi(c/\sigma_1)$ and $\phi'\left(-c/\sigma_1\right) = -\left(-c/\sigma_1\right)\phi\left(-c/\sigma_1\right) = \phi(c/\sigma_1)c/\sigma_1$ by the properties of the Gaussian pdf and cdf (in particular, note that $\Phi'\left(x\right) = \phi\left(x\right)$, $\phi\left(x\right) = \phi\left(-x\right)$ and $\phi'\left(x\right) = -x\phi\left(x\right)$). Hence,

$$I_1 = \left(\gamma_E - \gamma_R\right)\sigma_1\phi(c/\sigma_1)c/\sigma_1^2\beta_{\bar{h}} = \left(\gamma_E - \gamma_R\right)\phi(c/\sigma_1)c/\sigma_1\beta_{\bar{h}},$$

and

$$I_2 = \left(\gamma_E - \gamma_R\right)\sigma_1^{-1}\phi\left(c/\sigma_1\right)v_{\bar{h}}.$$

Thus,

$$CMR_1\left(\bar{h}\right) = \left\{\gamma_R + \left(\gamma_E - \gamma_R\right)\Phi(-c/\sigma_1)\right\}\beta_{\bar{h}} + \left(\gamma_E - \gamma_R\right)\phi\left(c/\sigma_1\right)\sigma_1^{-1}\left(c\beta_{\bar{h}} + v_{\bar{h}}\right).$$

■

Proof of Proposition 3.4. The result for $h = 0$ is immediate, so we focus on $h \geq 1$. For any such value of h, we can show that

$$b_h\left(\bar{h}\right) = \frac{E\left(y_{t+h}\varepsilon_{1t}|H_{t-1} = \bar{h}\right)}{E\left(\varepsilon_{1t}^2|H_{t-1} = \bar{h}\right)} = \left(\bar{\gamma}\right)^{h-1} b_1\left(\bar{h}\right),$$

using the fact that γ_t is i.i.d. since it is a function of ε_{1t}. Thus, we focus on deriving $b_1\left(\bar{h}\right) = \frac{E\left(y_{t+1}\varepsilon_{1t}|H_{t-1}=\bar{h}\right)}{E\left(\varepsilon_{1t}^2|H_{t-1}=\bar{h}\right)}$. Note that the denominator of $b_1\left(\bar{h}\right)$ is equal to σ_1^2 under our assumptions, so it is sufficient to derive $E\left(y_{t+1}\varepsilon_{1t}|H_{t-1} = \bar{h}\right)$. Replacing y_{t+1} by equation (3) in the main text, we write

$$E\left(y_{t+1}\varepsilon_{1t}|H_{t-1} = \bar{h}\right) = E\left(\left(\beta_t\varepsilon_{1t+1} + \gamma_t y_t + \varepsilon_{2t+1}\right)\varepsilon_{1t}|H_{t-1} = \bar{h}\right) = E(\gamma_t y_t\varepsilon_{1t}|H_{t-1} = \bar{h}),$$

since $E\left(\beta_t \varepsilon_{1t+1}\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=E\left(\varepsilon_{2t+1}\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=0$. But since $\gamma_t=\gamma_R+(\gamma_E-\gamma_R)\,H_t$,

$E(\gamma_t y_t \varepsilon_{1t}|H_{t-1}=\bar{h})=(\gamma_E-\gamma_R)\,E\left(H_t y_t \varepsilon_{1t}|H_{t-1}=\bar{h}\right)+\gamma_R E\left(y_t\varepsilon_{1t}|H_{t-1}=\bar{h}\right)\equiv(\gamma_E-\gamma_R)\,A_1+\gamma_R A_2.$

It follows that

$$\begin{aligned}
A_1 &\equiv E\left(\varepsilon_{1t}H_t y_t|H_{t-1}=\bar{h}\right)\\
&= E\left(\varepsilon_{1t}H_t\left(\beta_{t-1}\varepsilon_{1t}+\gamma_{t-1}y_{t-1}+\varepsilon_{2t}\right)|H_{t-1}=\bar{h}\right)\\
&= E\left(\varepsilon_{1t}^2 H_t|H_{t-1}=\bar{h}\right)\beta_{\bar{h}}+E\left(\varepsilon_{1t}H_t\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h}\right)+E\left(\varepsilon_{1t}\varepsilon_{2t}H_t|H_{t-1}=\bar{h}\right)\\
&= E\left(\varepsilon_{1t}^2 H_t\right)\beta_{\bar{h}}+E\left(\varepsilon_{1t}H_t\right)\underbrace{E\left(\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h}\right)}_{\equiv v_{\bar{h}}}+0,
\end{aligned}$$

where $E\left(\varepsilon_{1t}\varepsilon_{2t}H_t|H_{t-1}=\bar{h}\right)=0$ by the fact that $\varepsilon_{1t}H_t$ is independent of ε_{2t} under Assumptions 1 and 3. Similarly, we can write $E\left(\varepsilon_{1t}H_t\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h}\right)=E\left(\varepsilon_{1t}H_t\right)v_{\bar{h}}$, where $v_{\bar{h}}\equiv E\left(V_{0t}|H_{t-1}=\bar{h}\right)=E(\gamma_{t-1}y_{t-1}|H_{t-1}=\bar{h})$. Next, we compute $E\left(\varepsilon_{1t}H_t\right)$ and $E\left(\varepsilon_{1t}^2H_t\right)$ using the fact that ε_{1t} is Gaussian. By definition of $H_t=1\left(\varepsilon_{1t}>c\right)$, and the truncated moments of the Gaussian distribution, we obtain that

$$E\left(\varepsilon_{1t}H_t\right)=\sigma_1 E\left(\varepsilon_{1t}/\sigma_1 1\left(\varepsilon_{1t}/\sigma_1>c/\sigma_1\right)\right)=\sigma_1\phi\left(c/\sigma_1\right).$$

Similarly,

$$E\left(\varepsilon_{1t}^2H_t\right)=E\left(\varepsilon_{1t}^2 1\left(\varepsilon_{1t}>c\right)\right)=\sigma_1^2[\Phi\left(-c/\sigma_1\right)+c/\sigma_1\phi\left(c/\sigma_1\right)].$$

Thus

$$\frac{A_1}{\sigma_1^2}=[\Phi\left(-c/\sigma_1\right)+c/\sigma_1\phi\left(c/\sigma_1\right)]\beta_{\bar{h}}+\sigma_1^{-1}\phi\left(c/\sigma_1\right)v_{\bar{h}}.$$

Since we can also show that

$$\frac{A_2}{\sigma_1^2}=\sigma_1^{-2}E\left(y_t\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=\sigma_1^{-2}E\left((\beta_{t-1}\varepsilon_{1t}+\gamma_{t-1}y_{t-1}+\varepsilon_{2t})\varepsilon_{1t}|H_{t-1}=\bar{h}\right)=\beta_{\bar{h}},$$

it follows that

$$\begin{aligned}
b_1\left(\bar{h}\right) &= (\gamma_E-\gamma_R)\frac{A_1}{\sigma_1^2}+\gamma_R\frac{A_2}{\sigma_1^2}\\
&= (\gamma_E-\gamma_R)\left\{[\Phi\left(-c/\sigma_1\right)+c/\sigma_1\phi\left(c/\sigma_1\right)]\beta_{\bar{h}}+\sigma_1^{-1}\phi\left(c/\sigma_1\right)v_{\bar{h}}\right\}+\gamma_R\beta_{\bar{h}}\\
&= \left\{\gamma_R\beta_{\bar{h}}+(\gamma_E-\gamma_R)\,\Phi\left(-c/\sigma_1\right)\right\}\beta_{\bar{h}}+(\gamma_E-\gamma_R)\,\sigma_1^{-1}\phi\left(c/\sigma_1\right)\left(c\beta_{\bar{h}}+v_{\bar{h}}\right)\\
&= CMR_1\left(\bar{h}\right).
\end{aligned}$$

■

B Generalization of Propositions 3.1 and 3.2

Here, we show that the results in Section 3.1 extend to a multivariate version of our model for $z_t = (x_t, y_t')'$ when H_t is exogenous.

B.1 Multivariate state-dependent structural VAR model

Let $z_t \equiv (x_t, y_t')'$ denote an $n \times 1$ vector of strictly stationary time series, where y_t is $k \times 1$ with $k = n - 1$. We consider a structural state-dependent VAR process of the form

$$C_{t-1} z_t = \mu_{t-1} + B_{t-1}(L) z_{t-1} + \varepsilon_t, \tag{5}$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}')'$ defines the vector of mutually independent structural shocks. Let

$$B_{t-1}(L) = B_{1,t-1} + B_{2,t-1}L + \ldots + B_{p,t-1}L^{p-1},$$

where p denotes the polynomial lag order. For later convenience, we partition $B_{t-1}(L)$ conformably with z_t as

$$B_{t-1}(L) = \begin{pmatrix} B_{11,t-1}(L) & B_{12,t-1}(L) \\ B_{21,t-1}(L) & B_{22,t-1}(L) \end{pmatrix}$$

where $\mathcal{A}_{ij}$ denotes the (i,j) block of any partitioned matrix $\mathcal{A}$.

All model coefficients evolve over time depending on the state of the economy. In particular, as in the main text, we let

$$\begin{aligned} \mu_{t-1} &= \mu_E H_{t-1} + \mu_R (1 - H_{t-1}), \\ C_{t-1} &= C_E H_{t-1} + C_R (1 - H_{t-1}), \text{ and} \\ B_{j,t-1} &= B_{jE} H_{t-1} + B_{jR} (1 - H_{t-1}) \text{ for } j = 1, \ldots, p, \end{aligned}$$

where H_{t-1} is a binary stationary time series that takes the value 1 if the economy is in expansion and 0 otherwise. To identify the conditional impulse response function of y_{t+h} to a shock in ε_{1t}, we assume that

$$C_{t-1} = \begin{pmatrix} 1 & 0 \\ -C_{21,t-1} & C_{22,t-1} \end{pmatrix}, \tag{6}$$

where $C_{21,t-1}$ is $k \times 1$ and $C_{22,t-1}$ is a $k \times k$ non-singular matrix whose diagonal elements are 1 by a standard normalization condition. Under these assumptions, x_t is predetermined with respect to y_t. Note that we do not restrict $C_{22,t-1}$ to be lower triangular, which allows C_{t-1} to be block recursive. Hence, the model is only partially identified in that only the responses to ε_{1t} are identified.

Model (5) covers several empirically relevant strategies for identifying the structural shock ε_{1t} (and the corresponding conditional response function for y_{t+h} with respect to ε_{1t}). One is the narrative approach to identification which uses information extraneous to the model to measure ε_{1t}, in which case $x_t = \varepsilon_{1t}$ (as in the main text). Alternatively, the structural shock ε_{1t} may be identified via an exclusion restriction that precludes x_t from responding contemporaneously to the structural shocks in the remaining variables of the system. In this case, the structural shock ε_{1t} is identified within the nonlinear structural VAR model by analogy to Blanchard and Perotti (2002), whose exogenous shocks to government spending (ε_{1t}) are identified by assuming that government spending (x_t) does not react within the period to shocks to output and tax revenues (y_t). Finally, note that our general model also accommodates the special case of x_t being an exogenous serially correlated observable variable, as in Alloza, Gonzalo and Sanz (2021).

The structural model for z_t can be written as

$$\begin{cases} x_t = \mu_{1,t-1} + B_{11,t-1}(L)\, x_{t-1} + B_{12,t-1}(L)\, y_{t-1} + \varepsilon_{1t} \\ \\ C_{22,t-1} y_t = \mu_{2,t-1} + C_{21,t-1} x_t + B_{21,t-1}(L)\, x_{t-1} + B_{22,t-1}(L)\, y_{t-1} + \varepsilon_{2t}. \end{cases} \tag{7}$$

Without further restrictions (such as postulating that $C_{22,t-1}$ is lower triangular), the parameters in the equations for y_t are not identified. However, the fact that ε_{1t} is identified suffices to identify the conditional response function of y_t to a one-time shock in ε_{1t}.

As in Section 3.1, we assume that H_{t-1} is a function only of q_t (and its lags), where q_t is assumed to be exogenous with respect to the structural shocks ε_{1t} and ε_{2t}. More specifically, to complete the model, we let

$$H_t = \eta\,(q_s : s \leq t)\,. \tag{8}$$

We make the following additional assumptions.

Assumption B.1 *$\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ are mutually independent structural shocks such that $\varepsilon_t \equiv (\varepsilon_{1t}, \varepsilon_{2t}')' \sim$ i.i.d.$(0, \Sigma)$, where Σ is a diagonal matrix with diagonal elements given by σ_i^2 for $i = 1, \ldots, n$. In addition, y_t is strictly stationary and ergodic.*

Assumption B.2 *$\{q_t\}$ is independent of $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$.*

Assumption B.1 is the generalization of Assumption 1 in Section 3.1 to the multivariate model where ε_{2t} is a $k \times 1$ vector. Assumption B.2 is the analogue of Assumption 2.

B.2 Conditional impulse response functions

In this section, we derive the analogue of Proposition 3.1 in the main text for the multivariate model considered in (7) and (8). We obtain this result by first deriving the potential outcomes $y_{t+h}(e)$ and then using these to obtain closed-form expressions for $CAR_h\left(\delta,\bar{h}\right)$ and $CMR_h\left(\delta,\bar{h}\right)$.

B.2.1 Potential outcomes

To derive the potential outcomes $y_{t+h}(e)$, we first obtain the reduced-form model corresponding to our structural model (7) (which is given by (5) with the identification restriction that x_t is predetermined with respect to ε_{1t}). Since C_{t-1} satisfies the identification condition (6), the inverse matrix of C_{t-1} exists and is given by

$$C_{t-1}^{-1}=\begin{pmatrix} 1 & 0 \\ C_{22,t-1}^{-1}C_{21,t-1} & C_{22,t-1}^{-1}\end{pmatrix}\equiv\begin{pmatrix} 1 & 0 \\ C_{t-1}^{21} & C_{t-1}^{22}\end{pmatrix},$$

where for any matrix $\mathcal{A}$, we let $\mathcal{A}^{ij}$ denote the block (i,j) of $\mathcal{A}^{-1}$.

Pre-multiplying (5) by C_{t-1}^{-1} yields

$$z_t=C_{t-1}^{-1}\mu_{t-1}+C_{t-1}^{-1}B_{t-1}(L)\,z_{t-1}+C_{t-1}^{-1}\varepsilon_t,$$

which we rewrite as

$$z_t=b_{t-1}+A_{t-1}(L)\,z_{t-1}+\eta_t, \tag{9}$$

where $\eta_t\equiv C_{t-1}^{-1}\varepsilon_t$, $b_{t-1}\equiv C_{t-1}^{-1}\mu_{t-1}$, and

$$A_{t-1}(L)\equiv C_{t-1}^{-1}B_{t-1}(L)=A_{1,t-1}+A_{2,t-1}L+\ldots+A_{p,t-1}L^{p-1},$$

with $A_{j,t-1}\equiv C_{t-1}^{-1}B_{j,t-1}$.

The potential outcome value of $y_{t+h}(e)$ (for any fixed e) can be obtained from the companion-form representation of the reduced-form model (9) by iteration, fixing $\varepsilon_{1t}=e$. Since only ε_{1t} is fixed at e, the following decomposition of the reduced-form errors η_t is useful:

$$\eta_t\equiv C_{t-1}^{-1}\varepsilon_t=\begin{pmatrix} 1 \\ C_{t-1}^{21}\end{pmatrix}\varepsilon_{1t}+\begin{pmatrix} 0 \\ C_{t-1}^{22}\end{pmatrix}\varepsilon_{2t}\equiv C_{t-1}^{-1}e_{1,n}\varepsilon_{1t}+C_{t-1}^{-1}I_{2:n}\varepsilon_{2t},$$

where $e_{1,n}\equiv(1,0')'$ is $n\times 1$ and $I_{2:n}$ is $k\times n$ and is equal to the $n\times n$ identity matrix with its first column removed:

$$I_{2:n}=\begin{pmatrix} e_{2,n} & \cdots & e_{n,n}\end{pmatrix}.$$

We let

$$\eta_t(e)=C_{t-1}^{-1}\begin{pmatrix} e \\ \varepsilon_{2t}\end{pmatrix}=C_{t-1}^{-1}e_{1,n}e+C_{t-1}^{-1}I_{2:n}\varepsilon_{2t}$$

denote the counterfactual value of η_t for $\varepsilon_{1t} = e$. Similarly, we denote by

$$z_t(e) = \begin{pmatrix} x_t(e) \\ y_t(e) \end{pmatrix}$$

the counterfactual values of x_t and y_t. With this notation, we can write the potential outcome analogue of (9) as

$$Z_t(e) = a_{t-1} + A_{t-1} Z_{t-1}(e) + \xi_t(e). \tag{10}$$

Here,

$$\underset{np\times 1}{Z_t}(e) = \left(z_t'(e), z_{t-1}'(e), \ldots, z_{t-p+1}'(e)\right)', \; \underset{np\times 1}{\xi_t}(e) = \left(\eta_t'(e), 0'\right)', \; \underset{np\times 1}{a_{t-1}} = \left(b_{t-1}', 0'\right)',$$

and

$$\underset{np\times np}{A_{t-1}} = \begin{pmatrix} A_{1,t-1} & A_{2,t-1} & \cdots & A_{p-1,t-1} & A_{p,t-1} \\ I_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{pmatrix}.$$

Note that a_{t-1} and A_{t-1} are not indexed by e because these matrices depend only on H_{t-1}, which does not change with e under the exogeneity assumption on H_t. To obtain $y_t(e)$ from $Z_t(e)$, let

$$\underset{k\times np}{\mathbb{S}_k} = \begin{pmatrix} 0_{k\times 1} & I_k & 0_{k\times n(p-1)} \end{pmatrix}$$

denote a $k \times np$ selection matrix (with $k = n-1$ equal to the number of variables in y_t) which selects the subvector y_t from the vector Z_t. With this notation,

$$y_t(e) = \mathbb{S}_k Z_t(e),$$

and, more generally, for any h,

$$y_{t+h}(e) = \mathbb{S}_k Z_{t+h}(e).$$

Note that for $k = 1$ (i.e., for a bivariate system with $n = 2$), $\mathbb{S}_k = e_{2,2p}'$, where $e_{2,2p} = (0, 1, 0')$ is a $2p \times 1$ vector whose only non-zero element is equal to 1 and occurs in position 2. More generally, we let $e_{j,m}$ denote an $m \times 1$ vector with 1 in position j and 0 elsewhere.

Next, we use the companion form (10) to obtain $y_{t+h}(e)$ for different values of h. Starting with $h = 0$, we set $Z_{t-1}(e) = Z_{t-1}$ since Z_{t-1} depends on values of z_t that occur prior to the shock in ε_{1t}. Hence, these values do not depend on e and it follows that

$$y_t(e) = \mathbb{S}_k Z_t(e) = \mathbb{S}_k a_{t-1} + \mathbb{S}_k A_{t-1} Z_{t-1} + \mathbb{S}_k \xi_t(e).$$

By the definition of $\xi_t(e)$, we can write

$$\xi_t(e) = \begin{pmatrix} \eta_t(e) \\ 0 \end{pmatrix} = \begin{pmatrix} C_{t-1}^{-1} e_{1,n} e + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \\ 0_{n(p-1)\times 1} \end{pmatrix} = e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} e + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \right).$$

Hence,

$$\begin{aligned} \mathbb{S}_k \xi_t(e) &= \mathbb{S}_k [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} e + C_{t-1}^{-1} I_{2:n} \varepsilon_{2t} \right)] \\ &= \mathbb{S}_k [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right) e] + \mathbb{S}_k [e_{1,p} \otimes (C_{t-1}^{-1} I_{2:n}) \varepsilon_{2t}]. \end{aligned}$$

This implies that

$$y_t(e) = \mathbb{S}_k [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + V_t,$$

where $V_t \equiv \mathbb{S}_k a_{t-1} + \mathbb{S}_k A_{t-1} Z_{t-1} + \mathbb{S}_k [e_{1,p} \otimes (C_{t-1}^{-1} I_{2:n}) \varepsilon_{2t}]$ is a function of $U_t \equiv \left(\varepsilon_{2t}', q_{t-1}, Z_{t-1}' \right)$. We can obtain $y_{t+h}(e)$ for larger values of h using a similar approach. In particular, for $h = 1$, we have that

$$Z_{t+1}(e) = a_t + A_t Z_t(e) + \xi_{t+1},$$

where $\xi_{t+1} = \left(\eta_{t+1}', 0' \right)' = \left(\left(C_t^{-1} \varepsilon_{t+1} \right)', 0' \right)'$ and a_t, A_t and C_t do not depend on e. This is true because the model coefficients depend on H_t, which is not a function of e when H_t is exogenous, and ε_{t+1} is independent of e since e is the fixed value of ε_{1t}. Thus,

$$\begin{aligned} y_{t+1}(e) &= \mathbb{S}_k Z_{t+1}(e) \\ &= \mathbb{S}_k a_t + \mathbb{S}_k A_t Z_t(e) + \mathbb{S}_k \xi_{t+1} \\ &= \mathbb{S}_k a_t + \mathbb{S}_k A_t (a_{t-1} + A_{t-1} Z_{t-1} + \xi_t(e)) + \mathbb{S}_k \xi_{t+1} \\ &= \mathbb{S}_k a_t + \mathbb{S}_k A_t a_{t-1} + \mathbb{S}_k A_t A_{t-1} Z_{t-1} + \mathbb{S}_k A_t \xi_t(e) + \mathbb{S}_k \xi_{t+1}, \end{aligned}$$

where $\xi_t(e) = [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + \mathbb{S}_k [e_{1,p} \otimes (C_{t-1}^{-1} I_{2:n}) \varepsilon_{2t}]$. Inserting $\xi_t(e)$ into the equation above and collecting the terms that not depend on e into V_{t+1} yields

$$y_{t+1}(e) = \mathbb{S}_k A_t [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + V_{t+1},$$

where V_{t+1} is a function of $U_{t+1} \equiv \left(\varepsilon_{t+1}, \varepsilon_{2t}', q_t, q_{t-1}, Z_{t-1}' \right)'$. This result shows that the potential outcome value $y_{t+1}(e)$ is linear in e, as in the main text. This result generalizes to any value of $h \geq 1$ as follows:

$$y_{t+h}(e) = \mathbb{S}_k A_{t+h-1} \cdots A_t [e_{1,p} \otimes \left(C_{t-1}^{-1} e_{1,n} \right)] e + V_{t+h} \equiv m_h(e, U_{t+h}), \tag{11}$$

where V_{t+h} depends on $U_{t+h} \equiv \left(\varepsilon_{t+h}, \ldots, \varepsilon_{t+1}, \varepsilon_{2t}', q_{t+h-1}, \ldots, q_t, q_{t-1}, Z_{t-1}' \right)'$.

Equation (11) defines the potential outcomes for the vector of dependent variables y_t. It represents a linear function of e under the assumption that $H_t = \eta(q_s : s \leq t)$ and q_s is strictly exogenous with respect to ε_{1t} and ε_{2t}.

B.2.2 Closed-form expressions for the conditional response functions

Next, we use (11) to generalize Proposition 3.1 to the multivariate state-dependent structural VAR model given in (7). For any e,

$$y_{t+h}(e+\delta) - y_{t+h}(e) = \mathbb{S}_k A_{t+h-1} \cdots A_t [e_{1,p} \otimes (C_{t-1}^{-1} e_{1,n})] \delta,$$

which implies that letting $e = \varepsilon_{1t}$, and taking the conditional expectation, conditionally on $H_{t-1} = \bar{h} \in \{0,1\}$,

$$\begin{aligned} CAR_h(\delta, \bar{h}) &\equiv E\left(y_{t+h}(\varepsilon_{1t}+\delta) - y_{t+h}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right) \\ &= \mathbb{S}_k E\left(A_{t+h-1} A_{t+h-2} \ldots A_t | H_{t-1} = \bar{h}\right) \left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right) \delta. \end{aligned}$$

We can also use (11) to obtain the conditional marginal response function for this model. Since $y_{t+h}(e)$ is a linear function of e, it follows that

$$y'_{t+h}(e) \equiv \frac{\partial}{\partial e} m_h(e, U_{t+h}) = \mathbb{S}_k A_{t+h-1} \cdots A_t [e_{1,p} \otimes (C_{t-1}^{-1} e_{1,n})].$$

This implies that

$$\begin{aligned} CMR_h(\bar{h}) &\equiv E\left(y'_{t+h}(\varepsilon_{1t}) | H_{t-1} = \bar{h}\right) \\ &= \mathbb{S}_k E\left(A_{t+h-1} A_{t+h-2} \ldots A_t | H_{t-1} = \bar{h}\right) \left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right) \\ &= CAR_h(1, \bar{h}), \end{aligned}$$

showing that the conditional marginal response function coincides with the conditional average response function $CAR_h(\delta, \bar{h})$ for a shock of size $\delta = 1$.

The following proposition summarizes these results and is the analogue of Proposition 3.1 for the multivariate model considered in (7). We let $C_{\bar{h}}^{-1} = C_E^{-1}$ if $\bar{h} = 1$ and $C_{\bar{h}}^{-1} = C_R^{-1}$ if $\bar{h} = 0$.

Proposition B.1 *Assume the structural process is (7) and (8) with $H_t = \eta(q_s : s \leq t)$. Under Assumptions B.1 and B.2 for $\bar{h} \in \{0,1\}$:*

(i) *For any fixed δ, $CAR_0(\delta, \bar{h}) = \mathbb{S}_k \left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right) \delta$, and for any $h \geq 1$,*

$$CAR_h(\delta, \bar{h}) = \mathbb{S}_k E\left(A_{t+h-1} A_{t+h-2} \ldots A_t | H_{t-1} = \bar{h}\right) \left(e_{1,p} \otimes C_{\bar{h}}^{-1} e_{1,n}\right) \delta.$$

(ii) *For any $h \geq 0$, $CMR_h(\bar{h}) = CAR_h(\delta, \bar{h})$.*

As in the simpler model considered in the main text, Proposition B.1 shows that when H_t depends only on $\{q_s : s \leq t\}$, i.e., when H_t is exogenous with respect to the structural shocks ε_t, the two definitions of the conditional impulse response function coincide. Next, we show that the state-dependent local projection estimator recovers asymptotically these two notions of conditional impulse response functions when H_t is exogenous.

B.3 Local projections estimands

A state-dependent LP regression is a direct regression of y_{t+h} onto a constant, x_t and Z_{t-1}, each interacted with H_{t-1} and $1-H_{t-1}$. The slope coefficients associated with $x_t H_{t-1}$ are usually interpreted as the CAR of y_{t+h}, conditionally on $H_{t-1} = 1$, whereas the slope coefficients associated with $x_t(1 - H_{t-1})$ are interpreted as the CAR of y_{t+h} when we condition on $H_{t-1} = 0$. The goal of this section is to derive the probability limits of these slope coefficients and show that they equal $CAR_h\left(\delta, \bar{h}\right)$ when $\delta = 1$, which is equal to the $CMR_h\left(\bar{h}\right)$ for $\bar{h} \in \{0, 1\}$.

Let $W_{t-1} \equiv (1, Z'_{t-1})'$ denote an $(np+1) \times 1$ vector of control variables which include a constant and p lags of z_t. A state-dependent LP for identifying the causal effect on y_{t+h} of a one-time shock in ε_{1t} of size $\delta = 1$ can be written as

$$y_{t+h} = b_h\left(1\right) x_t H_{t-1} + \Pi_{E,h} W_{t-1} H_{t-1} + b_h\left(0\right) x_t (1 - H_{t-1}) + \Pi_{R,h} W_{t-1} (1 - H_{t-1}) + v_{t+h}, \qquad (12)$$

where the $k \times 1$ vectors $b_h\left(1\right)$ and $b_h\left(0\right)$ contain the main parameters of interest. The LP regression for variable $y_{j,t+h}$ is

$$y_{j,t+h} = b_{h,j}\left(1\right) x_t H_{t-1} + \pi'_{E,j,h} W_{t-1} H_{t-1} + b_{h,j}\left(0\right) x_t (1 - H_{t-1}) + \pi'_{R,j,h} W_{t-1} (1 - H_{t-1}) + v_{j,t+h}, \; (13)$$

where $j = 2, \ldots, n$. The scalar coefficients $b_{h,j}\left(1\right)$ and $b_{h,j}\left(0\right)$ are the $(j-1)^{th}$ elements of $b_h\left(1\right)$ and $b_h\left(0\right)$, respectively. Similarly, $\pi'_{E,j,h}$ and $\pi'_{R,j,h}$ are the corresponding rows of $\Pi_{E,h}$ and $\Pi_{R,h}$.

Since H_t is observed, the coefficients in the multivariate state-dependent LP regression (12) can be obtained by running a multivariate LS regression of y_{t+h} onto $x_t H_{t-1}$, $W_{t-1} H_{t-1}$, $x_t\left(1 - H_{t-1}\right)$ and $W_{t-1}\left(1 - H_{t-1}\right)$. Note that this is equivalent to running a regression of $y_{j,t+h}$ onto $x_t H_{t-1}$, $W_{t-1} H_{t-1}$, $x_t\left(1 - H_{t-1}\right)$ and $W_{t-1}\left(1 - H_{t-1}\right)$, for each $j = 2, \ldots, n$. Put differently, the multivariate LS regression (12) is equivalent to the k univariate OLS regressions (13), equation-by-equation.

Let $\hat{b}_h\left(1\right)$ and $\hat{b}_h\left(0\right)$ denote the LS estimators of $b_h\left(1\right)$ and $b_h\left(0\right)$ in (12) based on a sample of size T given by $\{y_{t+h}, x_t, Z_{t-1}, H_{t-1} : t = 1, \ldots, T\}$. We can estimate each of these vectors separately, by restricting the sample to $H_{t-1} = 1$ and $H_{t-1} = 0$, respectively. For instance, $\hat{b}_h\left(1\right)$ can be obtained from a regression of y_{t+h} on $x_t H_{t-1}$ and $W_{t-1} H_{t-1}$ (omitting $x_t\left(1 - H_{t-1}\right)$ and $W_{t-1}\left(1 - H_{t-1}\right)$ in the regression). This follows because $H_{t-1}(1-H_{t-1})=0$ for all t. Similarly, we can obtain $\hat{b}_h(0)$ from a regression of y_{t+h} on $x_t(1-H_{t-1})$ and $W_{t-1}(1-H_{t-1})$ (omitting $x_t H_{t-1}$ and $W_{t-1}H_{t-1}$ in this regression).

Our next result generalizes Proposition 3.2. to the multivariate structural VAR model given in (7) and (8).

Proposition B.2 *Consider the structural process (7) and (8) with $H_t=\eta(q_s : s\leq t)$. If Assumptions B.1 and B.2 hold, then for $\bar{h}\in\{0,1\}$,*

$$b_h\left(\bar{h}\right)\equiv p\lim_{T\to\infty}\hat{b}_h\left(\bar{h}\right)=CMR_h\left(\bar{h}\right)=CAR_h\left(1,\bar{h}\right),$$

where $CAR_h\left(1,\bar{h}\right)$ is the conditional average response function in Definition 1 with $\delta=1$.

B.4 Proofs of Propositions B.1 and B.2

Proof of Proposition B.1. The proof for $h=0$ and $h=1$ is in the text. We omit the proof for general h since it follows from similar arguments.

Proof of Proposition B.2. We focus on $\bar{h}=1$. To define $\hat{b}_h(1)$, let

$$\underset{T\times k}{Y}=\begin{pmatrix} y'_{1+h}\\ \vdots \\ y'_{T+h}\end{pmatrix},\quad \underset{T\times 1}{X_1}=\begin{pmatrix} x_1H_0\\ \vdots \\ x_TH_{T-1}\end{pmatrix},\quad\text{and}\quad \underset{T\times(np+1)}{X_2}=\begin{pmatrix} W'_0H_0\\ \vdots \\ W'_{T-1}H_{T-1}\end{pmatrix},$$

and define $M_2=I_T-X_2\left(X'_2X_2\right)^{-1}X'_2$.

By the Frisch-Waugh-Lovell (FWL) Theorem, $\hat{b}_h(1)'=\left(X'_1M_2X_1\right)^{-1}X'_1M_2Y$, or

$$\hat{b}_h(1)=T^{-1}(Y'M_2X_1)\left(T^{-1}X'_1M_2X_1\right)^{-1}\equiv\hat{Q}_{1y.2,h}\hat{Q}^{-1}_{11.2}.$$

A similar expression holds for $\hat{b}_h(0)$ with the difference that the regressors x_t and W_{t-1} are interacted with $1-H_{t-1}$ rather than H_{t-1}.

Our goal is to derive the probability limit of $\hat{b}_h(1)$ (and $\hat{b}_h(0)$) as $T\to\infty$. We can write

$$\begin{aligned}\hat{Q}_{11.2} &= T^{-1}X'_1X_1-T^{-1}X'_1X_2\left(T^{-1}X'_2X_2\right)^{-1}T^{-1}X'_2X_1,\ \text{and}\\ \hat{Q}_{1y.2,h} &= T^{-1}Y'X_1-T^{-1}Y'X_2\left(T^{-1}X'_2X_2\right)^{-1}T^{-1}X'_2X_1.\end{aligned}$$

If a law of large numbers applies to each term[1],

$$\begin{aligned}\hat{Q}_{11.2} \xrightarrow{p} Q_{11.2} &\equiv E\left(x_t^2 H_{t-1}\right) - E\left(x_t H_{t-1} W_{t-1}'\right)\left[E\left(W_{t-1}W_{t-1}'H_{t-1}\right)\right]^{-1} E\left(W_{t-1}H_{t-1}x_t\right), \text{ and}\\ \hat{Q}_{1y.2,h} \xrightarrow{p} Q_{1y.2,h} &\equiv E\left(y_{t+h}x_t H_{t-1}\right) - E\left(y_{t+h} H_{t-1} W_{t-1}'\right)\left[E\left(W_{t-1}W_{t-1}'H_{t-1}\right)\right]^{-1} E\left(W_{t-1}H_{t-1}x_t\right).\end{aligned}$$

We distinguish two cases: (i) $x_t = \varepsilon_{1t}$, and (ii) $x_t = \mu_{1,t-1} + B_{11,t-1}(L)\, x_{t-1} + B_{12,t-1}(L)\, y_{t-1} + \varepsilon_{1t} = \alpha_{t-1}' W_{t-1} + \varepsilon_{1t}$ (where α_{t-1} is a state-dependent vector that collects the coefficients of $\mu_{1,t-1}$, $B_{11,t-1}(L)$ and $B_{12,t-1}(L)$).

In case (i), it is easy to see that $E\left(x_t H_{t-1} W_{t-1}'\right) = 0$ under the assumption that $x_t = \varepsilon_{1t}$ is i.i.d. and independent of ε_{2t}. Thus,

$$Q_{11.2} = E\left(x_t^2 H_{t-1}\right) \text{ and } Q_{1y.2,h} = E\left(y_{t+h}x_t H_{t-1}\right),$$

implying that[2]

$$\hat{b}_h(1) \xrightarrow{p} b_h(1) \equiv E\left(y_{t+h}x_t H_{t-1}\right)\left[E\left(x_t^2 H_{t-1}\right)\right]^{-1} = E\left(y_{t+h}x_t | H_{t-1}=1\right)\left[E\left(x_t^2 | H_{t-1}=1\right)\right]^{-1}.$$

In case (ii), we can show that

$$\begin{aligned}Q_{11.2} &= E\left(\varepsilon_{1t}^2 H_{t-1}\right) = \Pr\left(H_{t-1}=1\right) E\left(\varepsilon_{1t}^2 | H_{t-1}=1\right) \text{ and}\\ Q_{1y.2,h} &= E\left(y_{t+h}\varepsilon_{1t} H_{t-1}\right) = \Pr\left(H_{t-1}=1\right) E\left(y_{t+h}\varepsilon_{1t} | H_{t-1}=1\right),\end{aligned}$$

implying that $\hat{b}_h(1) = E\left(y_{t+h}\varepsilon_{1t} | H_{t-1}=1\right)\left[E\left(\varepsilon_{1t}^2 | H_{t-1}=1\right)\right]^{-1}$. Heuristically, this follows because by the FWL theorem, and conditioning on $H_{t-1}=1$, the slope coefficient associated with x_t from regressing y_{t+h} on x_t and W_{t-1} can be obtained in two steps. First, we regress x_t on W_{t-1} (interacted with H_{t-1}) and obtain the residual. Under our identification condition, this is ε_{1t}. Then, we regress y_{t+h} on ε_{1t} (interacted with H_{t-1}). More specifically, note that

$$E\left(x_t H_{t-1} W_{t-1}'\right) = E\left(\alpha_{t-1}' W_{t-1} W_{t-1}' H_{t-1}\right) + E\left(\varepsilon_{1t} H_{t-1} W_{t-1}'\right) = E\left(\alpha_{t-1}' W_{t-1} W_{t-1}' H_{t-1}\right),$$

[1] This follows under the assumption that z_t is strictly stationary and ergodic and that the usual moment and rank conditions on the regressors are satisfied. We leave these as implicit high level assumptions since our focus here is on the conditions that H_t needs to satisfy in order for the LP estimator to be consistent. Kole and van Dijk (2021) (and references therein) provide primitive conditions for stationarity and ergodicity of a Markov Switching SVAR model when the states H_t are assumed to be a first-order exogenous Markov process. Deriving analogous primitive conditions for our setting, when the process for the exogenous H_t is not specified, is beyond the scope of this paper.

[2] This result is consistent with the fact that when x_t is a directly observed shock we can simply regress y_{t+h} onto $x_t H_{t-1}$ to obtain a consistent estimator of $b_{E,h}$. When $x_t = \varepsilon_{1t}$, adding the controls $W_{t-1}H_{t-1}$ is not required for consistency, but can be important for efficiency.

since $E\left(\varepsilon_{1t}H_{t-1}W'_{t-1}\right)=0$ by Assumption B.1. It follows that

$$E\left(x_tH_{t-1}W'_{t-1}\right)=\alpha'_E E\left(W_{t-1}W'_{t-1}|H_{t-1}=1\right)\Pr\left(H_{t-1}=1\right).$$

Hence, the term $E\left(x_tH_{t-1}W'_{t-1}\right)\left[E\left(W_{t-1}W'_{t-1}H_{t-1}\right)\right]^{-1}E\left(W_{t-1}H_{t-1}x_t\right)$ equals

$$\begin{aligned}
& \alpha'_E E\left(W_{t-1}W'_{t-1}|H_{t-1}=1\right)\left[E\left(W_{t-1}W'_{t-1}|H_{t-1}=1\right)\right]^{-1}E\left(W_{t-1}W'_{t-1}|H_{t-1}=1\right)\alpha_E\Pr\left(H_{t-1}=1\right)\\
= \; & \alpha'_E E\left(W_{t-1}W'_{t-1}|H_{t-1}=1\right)\alpha_E\Pr\left(H_{t-1}=1\right)\\
= \; & E\left(\alpha'_{t-1}W_{t-1}W'_{t-1}\alpha_{t-1}|H_{t-1}=1\right)\Pr\left(H_{t-1}=1\right).
\end{aligned}$$

Since $x_t^2=\left(\alpha'_{t-1}W_{t-1}+\varepsilon_{1t}\right)^2=\alpha'_{t-1}W_{t-1}W'_{t-1}\alpha_{t-1}+2\alpha'_{t-1}W_{t-1}\varepsilon_{1t}+\varepsilon_{1t}^2$, where the second term has a conditional mean of zero, it follows that

$$Q_{11.2}=\Pr\left(H_{t-1}=1\right)E\left(\varepsilon_{1t}^2|H_{t-1}=1\right).$$

One can use similar arguments to show that

$$Q_{1y.2,h}=\Pr\left(H_{t-1}=1\right)E\left(y_{t+h}\varepsilon_{1t}|H_{t-1}=1\right).$$

Thus, both in cases (i) and (ii), we conclude that

$$\hat{b}_h\left(1\right)\xrightarrow{p}b_h\left(1\right)=E\left(y_{t+h}\varepsilon_{1t}|H_{t-1}=1\right)\left[E\left(\varepsilon_{1t}^2|H_{t-1}=1\right)\right]^{-1}\equiv\mathcal{N}_h\mathcal{D},$$

where $\mathcal{N}_h$ stands for numerator and $\mathcal{D}$ is the denominator. Next, we express $\mathcal{N}_h$ and $\mathcal{D}$ in terms of the model parameters. To evaluate $\mathcal{N}_h$, we use the fact that for any h, $y_{t+h}=\mathbb{S}_kZ_{t+h}$, where Z_{t+h} is obtained from the companion-form representation of the model given by (10).

Consider first $h=0$. Then

$$Z_t=a_{t-1}+A_{t-1}Z_{t-1}+\xi_t,$$

where

$$\xi_t=\begin{pmatrix}\eta_t\\ 0\end{pmatrix}=\begin{pmatrix}C_{t-1}^{-1}e_{1,n}\varepsilon_{1t}+C_{t-1}^{-1}I_{2:n}\varepsilon_{2t}\\ 0\end{pmatrix}=(e_{1,p}\otimes C_{t-1}^{-1}e_{1,n})\varepsilon_{1t}+e_{1,p}\otimes C_{t-1}^{-1}I_{2:n}\varepsilon_{2t},$$

given that $\eta_t=C_{t-1}^{-1}\varepsilon_t$ and $\varepsilon_t=C_{t-1}^{-1}e_{1,n}\varepsilon_{1t}+C_{t-1}^{-1}I_{2:n}\varepsilon_{2t}$, where $e_{1,n}$ and $I_{2:n}$ are as defined in Section B.2. Hence,

$$y_t=\mathbb{S}_kZ_t=\mathbb{S}_k(e_{1,p}\otimes C_{t-1}^{-1}e_{1,n})\varepsilon_{1t}+\mathbb{S}_k(a_{t-1}+A_{t-1}Z_{t-1})+\mathbb{S}_k(e_{1,p}\otimes C_{t-1}^{-1}I_{2:n}\varepsilon_{2t}). \qquad (14)$$

Using this decomposition of y_t, we can write $\mathcal{N}_0 = E\left(y_t \varepsilon_{1t} | H_{t-1} = 1\right) = \mathcal{N}_{0,1} + \mathcal{N}_{0,2} + \mathcal{N}_{0,3}$, where

$$\begin{aligned}
\mathcal{N}_{0,1} &= E[\mathbb{S}_k(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n})\varepsilon_{1t}^2 | H_{t-1} = 1], \\
\mathcal{N}_{0,2} &= E[\mathbb{S}_k(a_{t-1} + A_{t-1} Z_{t-1})\varepsilon_{1t} | H_{t-1} = 1], \text{ and} \\
\mathcal{N}_{0,3} &= E[\mathbb{S}_k(e_{1,p} \otimes C_{t-1}^{-1} I_{2:n} \varepsilon_{2t})\varepsilon_{1t} | H_{t-1} = 1].
\end{aligned}$$

Under Assumption 1 and applying repeatedly the law of iterated expectations (LIE), it can be shown that $\mathcal{N}_{0,2} = \mathcal{N}_{0,3} = 0$, implying that $\mathcal{N}_0 \equiv E\left(y_t \varepsilon_{1t} | H_{t-1} = 1\right) = \mathcal{N}_{0,1}$. Thus,

$$\mathcal{N}_0 = \mathbb{S}_k(e_{1,p} \otimes C_E^{-1} e_{1,n}) E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right).$$

Since $b_h(1) \equiv \mathcal{N}_0 \mathcal{D}$, for $h = 0$, where $\mathcal{D} \equiv [E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)]^{-1}$, this implies the result. A similar argument shows that

$$\hat{b}_h(0) \xrightarrow{p} b_h(0) = \mathbb{S}_k(e_{1,p} \otimes C_R^{-1} e_{1,n}) \text{ for } h = 0.$$

Next, we consider $h = 1$. Now,

$$\hat{b}_h(1) \xrightarrow{p} b_h(1) \equiv E\left(y_{t+1}\varepsilon_{1t} | H_{t-1} = 1\right)\left[E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)\right]^{-1} \equiv \mathcal{N}_1 \mathcal{D} \text{ when } h = 1.$$

To obtain $\mathcal{N}_1$, we can use the fact that

$$\begin{aligned}
y_{t+1} &= \mathbb{S}_k Z_{t+1} = \mathbb{S}_k(a_t + A_t Z_t + \xi_{t+1}) \\
&= \mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1} + \xi_t) + \xi_{t+1}) \\
&= \mathbb{S}_k A_t \xi_t + \mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1})) + \mathbb{S}_k \xi_{t+1}, \qquad (15)
\end{aligned}$$

where $\xi_s = (e_{1,p} \otimes C_{s-1}^{-1} e_{1,n})\varepsilon_{1s} + e_{1,p} \otimes C_{s-1}^{-1} I_{2:n} \varepsilon_{2s}$ for $s = t, t+1$. This implies that $\mathcal{N}_1 \equiv E\left(y_{t+1}\varepsilon_{1t} | H_{t-1} = 1\right) = \mathcal{N}_{1,1} + \mathcal{N}_{1,2} + \mathcal{N}_{1,3}$, where

$$\begin{aligned}
\mathcal{N}_{1,1} &= E(\mathbb{S}_k A_t \xi_t \varepsilon_{1t} | H_{t-1} = 1), \\
\mathcal{N}_{1,2} &= E[\mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1}))\varepsilon_{1t} | H_{t-1} = 1], \text{ and} \\
\mathcal{N}_{1,3} &= E[\mathbb{S}_k \xi_{t+1} \varepsilon_{1t} | H_{t-1} = 1].
\end{aligned}$$

Given the definition of ξ_{t+1}, we can easily see that $\mathcal{N}_{1,3} = 0$ by Assumption B.1, since it implies that $E\left(\xi_{t+1} | \mathcal{F}^t\right) = 0$. To conclude that $\mathcal{N}_{1,2} = 0$, we use the exogeneity condition on H_t, i.e. the fact that $H_t = \eta\left(q_s : s \leq t\right)$ with q_s satisfying Assumption B.2. Under these assumptions, H_t and ε_{1t} are mutually independent, implying that by the LIE, we can write

$$\mathcal{N}_{1,2} = E[\mathbb{S}_k(a_t + A_t(a_{t-1} + A_{t-1} Z_{t-1})) E\left(\varepsilon_{1t} | \mathcal{F}^{t-1}, H_t\right) | H_{t-1} = 1],$$

where $\mathcal{F}^{t-1} = \sigma(z_{t-1}, H_{t-1}, z_{t-2}, H_{t-2}, \ldots)$. Since $E\left(\varepsilon_{1t}|\mathcal{F}^{t-1}, H_t\right) = E(\varepsilon_{1t}) = 0$, we obtain that $\mathcal{N}_{1,2} = 0$. Hence, $\mathcal{N}_1 = \mathcal{N}_{1,1}$. The result follows because we can show that

$$\mathcal{N}_{1,1} = E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n})\varepsilon_{1t}^2 | H_{t-1} = 1],$$

under Assumption B.1 and B.2. More specifically, using the definition of ξ_t, $\mathcal{N}_{1,1}$ can be decomposed as follows:

$$\mathcal{N}_{1,1} = E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n})\varepsilon_{1t}^2 | H_{t-1} = 1] + E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} I_{2:n}\varepsilon_{2t}\varepsilon_{1t}) | H_{t-1} = 1],$$

where $E\left(\varepsilon_{1t}\varepsilon_{2t}|H_t, \mathcal{F}^{t-1}\right) = E(\varepsilon_{1t}\varepsilon_{2t}) = 0$ under our assumptions. This implies that

$$b_h(1) = \frac{E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n})\varepsilon_{1t}^2 | H_{t-1} = 1]}{E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)}.$$

The result follows because the numerator simplifies to $E[\mathbb{S}_k A_t(e_{1,p} \otimes C_{t-1}^{-1} e_{1,n}) | H_{t-1} = 1][E\left(\varepsilon_{1t}^2 | H_{t-1} = 1\right)]$ under the assumption that ε_{1t} is i.i.d.$\left(0, \sigma_1^2\right)$. A similar result holds for $b_h(0)$ when $h = 1$. The proof for other values of h follows from similar arguments.

C Parameters for the data generating process in Section 5

The data generating process in Section 5 uses the following parameter values obtained by fitting the model to the quarterly data used in Ramey and Zubairy (2018), assuming that a recession corresponds to periods when unemployment is above the historical mean:

$$C_E = \begin{bmatrix} 1 & 0 & 0 \\ -0.0097 & 1 & 0 \\ 0.0056 & 0.0371 & 1 \end{bmatrix}, C_R = \begin{bmatrix} 1 & 0 & 0 \\ -0.0495 & 1 & 0 \\ -0.0510 & -0.2134 & 1 \end{bmatrix}, k_E = \begin{bmatrix} 0 \\ 0.0034 \\ 0.0177 \end{bmatrix}, k_R = \begin{bmatrix} 0 \\ 0.0145 \\ 0.1007 \end{bmatrix},$$

$$A_{E,1} = C_E^{-1} B_{E,1} = \begin{bmatrix} -0.1741 & 0 & 0 \\ 0.0317 & 0.8185 & -0.0437 \\ -0.0586 & 0.7540 & 1.4140 \end{bmatrix}, A_{E,2} = \begin{bmatrix} 0.4266 & 0 & 0 \\ 0.1107 & -0.0105 & 0.1177 \\ 0.0296 & -0.7467 & -0.4706 \end{bmatrix},$$

$$A_{E,3} = \begin{bmatrix} 0.4065 & 0 & 0 \\ 0.0889 & 0.2965 & -0.1358 \\ 0.0168 & -0.3586 & 0.0918 \end{bmatrix}, A_{E,4} = \begin{bmatrix} 0.3633 & 0 & 0 \\ 0.0774 & -0.1165 & 0.0595 \\ 0.0535 & 0.3428 & -0.0505 \end{bmatrix},$$

$$A_{R,1} = \begin{bmatrix} 0.2952 & 0 & 0 \\ 0.0088 & 1.6449 & 0.1237 \\ 0.0098 & 0.0450 & 1.4823 \end{bmatrix}, A_{R,2} = \begin{bmatrix} -0.0854 & 0 & 0 \\ 0.0463 & -0.8551 & -0.1995 \\ -0.0051 & -0.0752 & -0.7047 \end{bmatrix},$$

$$A_{R,3} = \begin{bmatrix} 0.1670 & 0 & 0 \\ 0.0107 & 0.2722 & 0.0245 \\ -0.0154 & 0.0911 & 0.2347 \end{bmatrix}, A_{R,4} = \begin{bmatrix} -0.0331 & 0 & 0 \\ -0.0019 & -0.0869 & 0.0410 \\ 0.0476 & -0.0333 & -0.1174 \end{bmatrix}.$$

D Additional simulation results

This appendix contains additional simulation results. Figures D.1 and D.2 report simulation results when $\gamma_E = 0.9$, $\gamma_R = -0.1$ in DGP 1 and DGP 2. Figures D.3 and D.4 report the cumulative government spending multiplier for $\delta \in \{-1, -5, -10\}$.

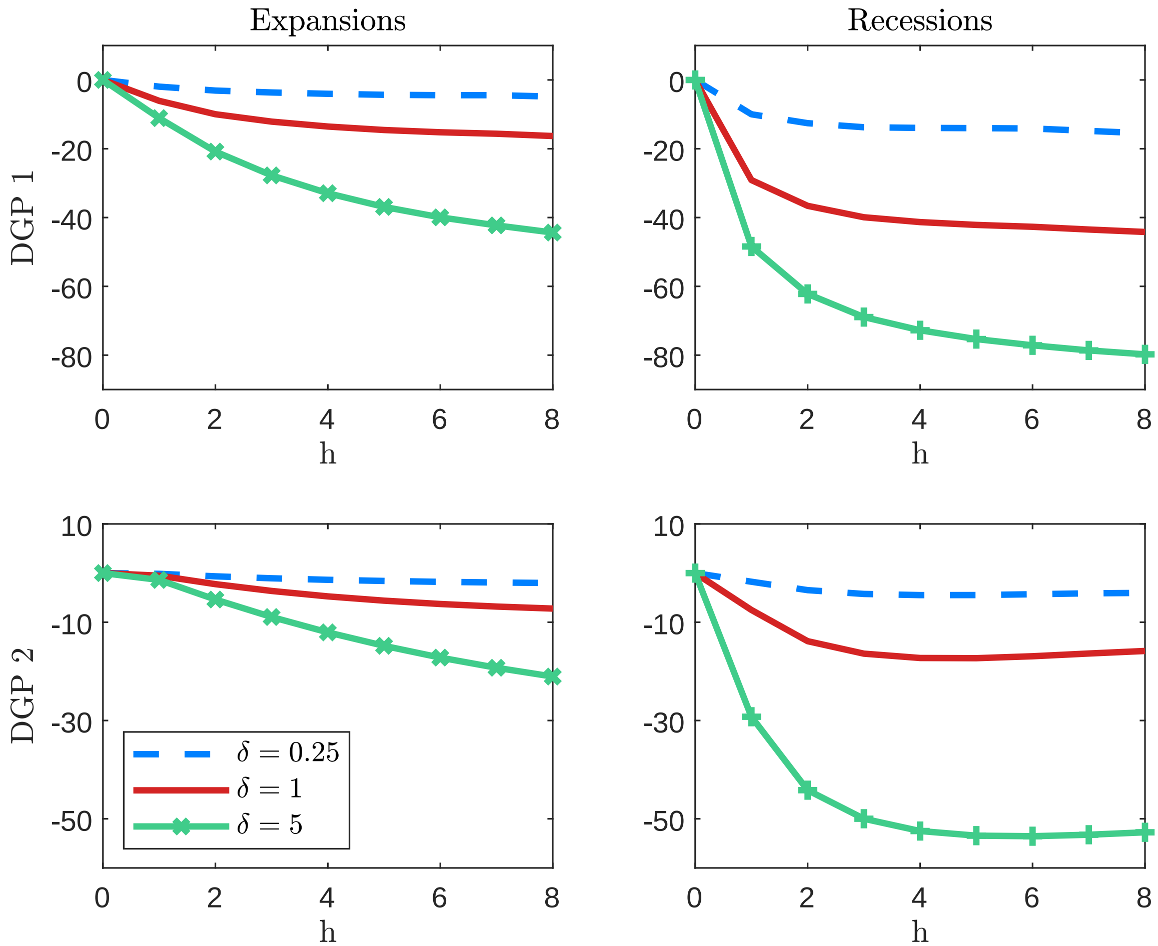

Figure D.1: Asymptotic bias of LP response when $H_t = 1\,(y_t > 0)$

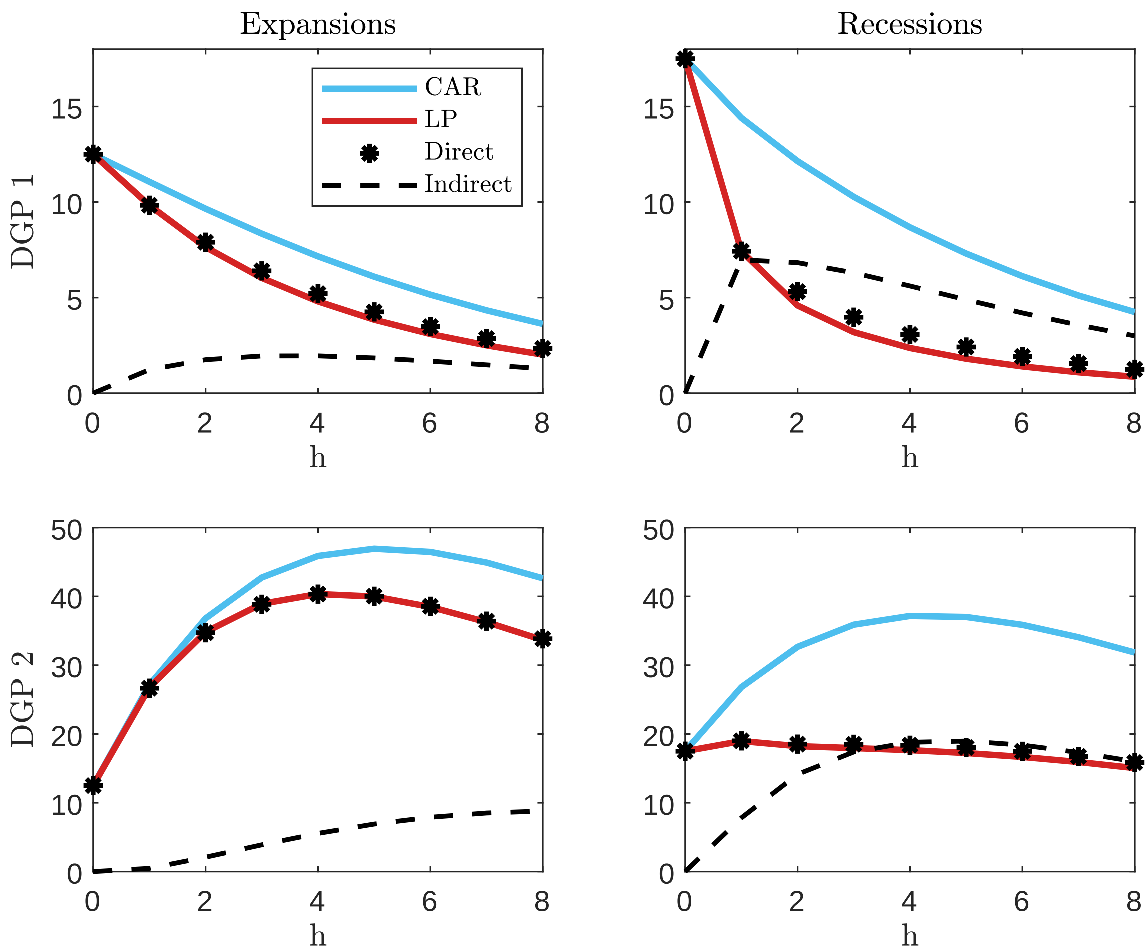

Figure D.2: LP response and decomposition of CAR when $H_t = 1\,(y_t > 0)$ and $\delta = 5$

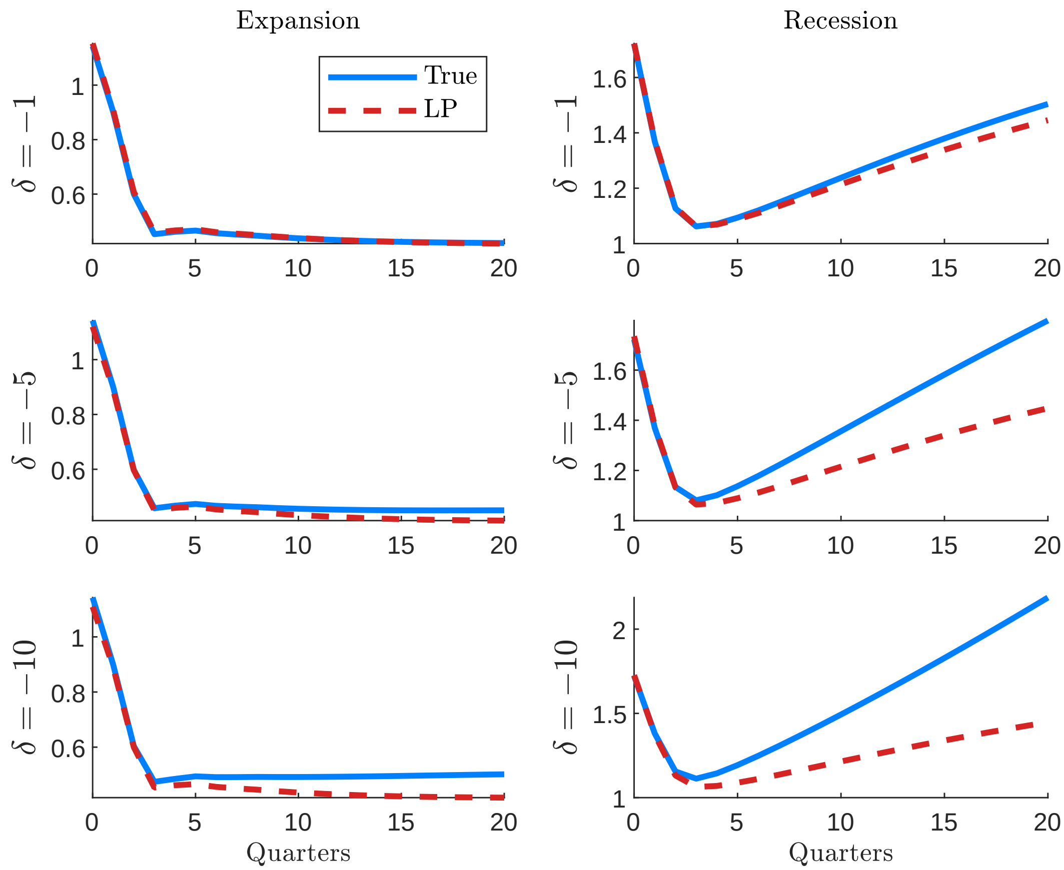

Figure D.3: Cumulative spending multiplier when $H_t = 1\,(y_t > 1)$

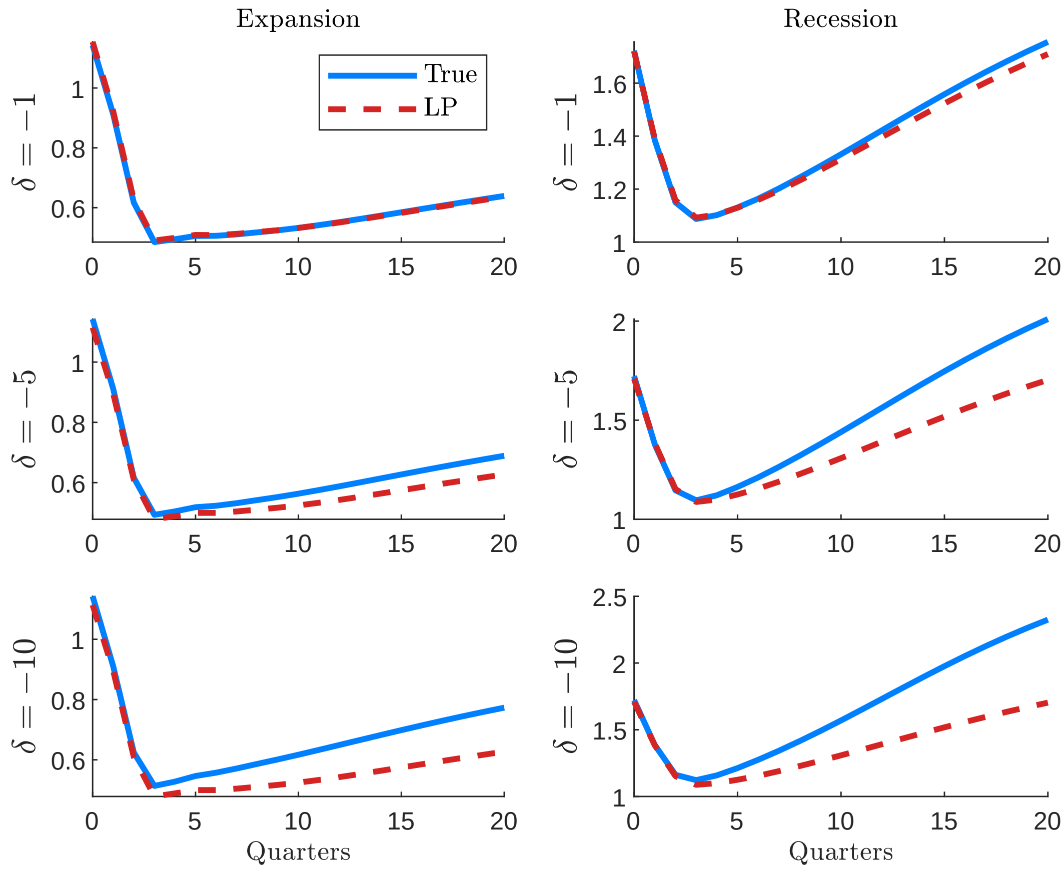

Figure D.4: Cumulative spending multiplier when $H_t = 1\,(y_t > MA(12))$

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