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Online Supplement to "Pooled Bewley Estimator of Long-Run Relationships in Dynamic Heterogenous Panels"

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"Pooled Bewley Estimator of Long Run Relationships in Dynamic Heterogenous Panels"

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This online supplement describes implementation of the Pooled Mean Group (PMG) estimator and its bias-corrected versions.

S-1 PMG estimator and its bias-corrected versions

Consider the same illustrative panel ARDL model as in the paper, namely the model given by equations (1)-(2). PMG estimator of the long-run coefficient β, as originally proposed by Pesaran, Shin and Smith (1999), is computed by solving the following equations iteratively:

$$\hat{\beta}_{PMG} = -\left(\sum_{i=1}^{n} \frac{\hat{\phi}_i^2}{\hat{\sigma}_i^2} \mathbf{x}_i' \mathbf{H}_{x,i} \mathbf{x}_i \right)^{-1} \sum_{i=1}^{n} \frac{\hat{\phi}_i^2}{\hat{\sigma}_i^2} \mathbf{x}_i' \mathbf{H}_{x,i} \left(\Delta \mathbf{y}_i - \hat{\phi}_i \mathbf{y}_{i,-1} \right), \tag{S.1}$$

$$\hat{\phi}_i = \left(\hat{\boldsymbol{\xi}}_i' \mathbf{H}_{x,i} \hat{\boldsymbol{\xi}}_i \right)^{-1} \hat{\boldsymbol{\xi}}_i' \mathbf{H}_{x,i} \Delta \mathbf{y}_i, \; i = 1, 2, ..., n, \tag{S.2}$$

and

$$\hat{\sigma}_i^2 = T^{-1} \left(\Delta \mathbf{y}_i - \hat{\phi}_i \hat{\boldsymbol{\xi}}_i \right)' \mathbf{H}_{x,i} \left(\Delta \mathbf{y}_i - \hat{\phi}_i \hat{\boldsymbol{\xi}}_i \right), \; i = 1, 2, ..., n, \tag{S.3}$$

where $\hat{\boldsymbol{\xi}}_i = \mathbf{y}_{i,-1} - \mathbf{x}_i \hat{\beta}_{PMG}$, $\mathbf{x}_i = (x_{i,1}, x_{i,2}, ..., x_{i,T})'$, $\Delta \mathbf{y}_i = \mathbf{y}_i - \mathbf{y}_{i,-1}$, $\mathbf{y}_i = (y_{i,1}, y_{i,2}, ..., y_{i,T})'$, $\mathbf{y}_{i,-1} = (y_{i,0}, y_{i,1}, ..., y_{i,T-1})'$, $\mathbf{H}_{x,i} = \mathbf{I}_T - \Delta \mathbf{x}_i \left(\Delta \mathbf{x}_i' \Delta \mathbf{x}_i \right)^{-1} \Delta \mathbf{x}_i'$, $\Delta \mathbf{x}_i = \mathbf{x}_i - \mathbf{x}_{i,-1}$, and $\mathbf{x}_{i,-1} = (x_{i,0}, x_{i,1}, ..., x_{i,T-1})'$.

To solve (S.1)-(S.3) iteratively, we set $\hat{\beta}_{PMG,(0)}$ to the pooled Engle-Granger estimator, and given the initial estimate $\hat{\beta}_{PMG,(0)}$, we compute $\hat{\boldsymbol{\xi}}_{i,(0)} = \mathbf{y}_{i,-1} - \mathbf{x}_i \hat{\beta}_{PMG,(0)}$, $\hat{\phi}_{i,(0)}$ and $\hat{\sigma}_{i,(0)}^2$, for $i = 1, 2, ..., n$ using (S.2)-(S.3). Next we compute $\hat{\beta}_{PMG,(1)}$ using (S.1) and given values $\hat{\phi}_{i,(0)}$ and

$\hat{\sigma}^2_{i,(0)}$. Then we iterate - for a given value of $\hat{\beta}_{PMG,(\ell)}$ we compute $\hat{\boldsymbol{\xi}}_{i,(\ell)}$, $\hat{\phi}_{i,(\ell)}$ and $\hat{\sigma}^2_{i,(\ell)}$; and for given values of $\hat{\phi}_{i,(\ell)}$ and $\hat{\sigma}^2_{i,(\ell)}$ we compute $\hat{\beta}_{PMG,(\ell+1)}$. If convergence is not achieved, we increase ℓ by one and repeat. We define convergence by $\left|\hat{\beta}_{PMG,(\ell+1)} - \hat{\beta}_{PMG,(\ell)}\right| < 10^{-4}$.[10]

Inference is conducted using equation (17) of Pesaran, Shin and Smith (1999). In particular,

$$T\sqrt{n}\left(\hat{\beta}_{PMG} - \beta_0\right) \sim N\left(0, \Omega_{PMG}\right),$$

where

$$\Omega_{PMG} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{\phi_{i,0}}{\sigma^2_{i,0}} r_{x_i,x_i}\right)^{-1}, \text{ and } r_{x_i,x_i} = plim_{T\to\infty} T^{-2}\mathbf{x}_i'\mathbf{H}_{x,i}\mathbf{x}_i.$$

Standard error of $\hat{\beta}_{PMG}$, denoted as $se\left(\hat{\beta}_{PMG}\right)$, is estimated as

$$\widehat{se}\left(\hat{\beta}_{PMG}\right) = T^{-1}n^{-1/2}\hat{\Omega}_{PMG},$$

where

$$\hat{\Omega}_{PMG} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{\phi}_{i,0}}{\hat{\sigma}^2_{i,0}} \hat{r}_{x_i,x_i}\right)^{-1} \text{ and } \hat{r}_{x_i,x_i} = T^{-2}\mathbf{x}_i'\mathbf{H}_{x,i}\mathbf{x}_i. \tag{S.4}$$

S-1.1 Simulation-based bias-corrected PMG

Similarly to the simulation-based bias-corrected PB estimator, we consider the following bias-corrected PMG estimator

$$\tilde{\beta}_{PMG} = \hat{\beta}_{PMG} - \hat{b}_{PMG}, \tag{S.5}$$

where $\hat{b}_{PMG}$ an estimate of the bias of PMG estimator obtained by the following stochastic simulation algorithm, which resembles the algorithm in Subsection 2.2.1.

1. Compute $\hat{\beta}_{PMG}$. Given PMG estimate $\hat{\beta}_{PMG}$, estimate the remaining unknown coefficients of (1)-(2) by least squares, and compute residuals $\hat{u}_{y,it}, \hat{u}_{x,it}$.

2. For each $r = 1, 2, ..., R$, generate new draws for $\hat{u}^{(r)}_{y,it} = a^{(r)}_{y,it}\hat{u}_{y,it}$, and $\hat{u}^{(r)}_{x,it} = a^{(r)}_{x,it}\hat{u}_{x,it}$, where $a^{(r)}_{y,it}$,$a^{(r)}_{x,it}$ are randomly drawn from Rademacher distribution (Davidson and Flachaire, 2008) namely

$$a^{(r)}_{h,it} = \begin{cases} -1, & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases},$$

for $h = y, x$. Given the estimated parameters of (1)-(2) from Step 1, and initial values y_{i1}, x_{i1} generate simulated data $y^{(r)}_{it}, x^{(r)}_{it}$ for $t = 2, 3, ..., T$ and $i = 1, 2, ..., n$. Using the generated data compute $\hat{\beta}^{(r)}_{PMG}$.

[10] If convergence does not occur within the first 500 iterations, we stop and report potential divergence. This event did not happen in any of the simulations in this paper. Convergence of the PMG procedure above is typically fast.

3. Compute $\hat{b}_{PMG} = \left[R^{-1}\sum_{r=1}^{R}\hat{\beta}_{PMG}^{(r)} - \hat{\beta}_{PMG}\right]$.

The above procedure can be iterated by using the bias-corrected estimate $\tilde{\beta}_{PMG}$ in Step 1, although this is not considered in this paper.

We conduct inference by using the $1-\alpha$ confidence interval $C_{1-\alpha}\left(\tilde{\beta}_{PMG}\right) = \tilde{\beta}_{PMG} \pm \hat{k}\widehat{se}\left(\hat{\beta}_{PMG}\right) = \tilde{\beta}_{PMG} \pm T^{-1}n^{-1/2}\hat{k}\hat{\Omega}_{PMG}$, where $\hat{k}$ is computed by stochastic simulation. In particular, $\hat{k}$ is the $1-\alpha$ percent quantile of $\left\{\left|t_{PMG}^{(r)}\right|\right\}_{r=1}^{R}$, where $t_{PMG}^{(r)} = \tilde{\beta}_{PMG}^{(r)}/\widehat{se}\left(\hat{\beta}_{PMG}^{(r)}\right) = T^{-1}n^{-1/2}\tilde{\beta}_{PMG}^{(r)}/\hat{\Omega}_{PMG}^{(r)}$, $\tilde{\beta}_{PMG}^{(r)} = \hat{\beta}_{PMG}^{(r)} - \hat{b}_{PMG}$ is the bias-corrected PMG estimate of β in the *r-th* draw of the simulated data in the algorithm above, and $\hat{\Omega}_{PMG}^{(r)}$ is computed as in (S.4), but using the simulated data.

S-1.2 Jackknife and combined bias-corrected PMG estimators

We consider similar jackknife bias correction for PMG estimator as for the PB estimator in Section 2.2. In particular,

$$\tilde{\beta}_{jk-PMG} = \tilde{\beta}_{jk-PMG}(\kappa) = \hat{\beta}_{PMG} - \kappa\left(\frac{\hat{\beta}_{PMG,a} + \hat{\beta}_{PMG,b}}{2} - \hat{\beta}_{PMG}\right),$$

where $\hat{\beta}_{PMG}$ is the full sample PMG estimator, $\hat{\beta}_{PMG,a}$and $\hat{\beta}_{PMG,b}$ are the first and the second half sub-sample PMG estimators, and κ is suitably chosen weighting parameter. Under our setup with I(1) variables, we need to correct $\hat{\beta}_{PMG}$ for its $O\left(T^{-2}\right)$ bias, which gives $\kappa = 1/3$.

We also consider a combined, simulation-based adaptive jackknife bias correction where $\kappa = \hat{\kappa}_{NT}$ is data-dependent and computed by stochastic simulation. Specifically, we consider

$$\hat{\kappa}_{PMG} = \frac{\hat{b}_{PMG}}{\hat{b}_{PMG,a,b} - \hat{b}_{PMG}}, \tag{S.6}$$

where $\hat{b}_{PMG} = R^{-1}\sum_{r=1}^{R}\hat{\beta}_{PMG}^{(r)} - \hat{\beta}$, and $\hat{b}_{PMG,a,b} = \left(\hat{b}_{PMG,a} + \hat{b}_{PMG,b}\right)/2$, $\hat{b}_{PMG,a} = R^{-1}\sum_{r=1}^{R}\hat{\beta}_{PMG,a}^{(r)} - \hat{\beta}_{PMG,a}$, $\hat{b}_{PMG,b} = R^{-1}\sum_{r=1}^{R}\hat{\beta}_{PMG,b}^{(r)} - \hat{\beta}_{PMG,b}$.

We conduct inference by using the $1-\alpha$ confidence interval $C_{1-\alpha}\left(\tilde{\beta}_{jk-PMG}\right) = \tilde{\beta}_{jk-PMG} \pm \hat{k}_{jk}\widehat{se}\left(\hat{\beta}_{PMG}\right) = \tilde{\beta}_{jk-PMG} \pm \hat{k}_{jk}T^{-1}n^{-1/2}\hat{\Omega}_{PMG}$, where $\hat{k}_{jk} = \hat{k}_{jk}(\kappa)$ is computed by stochastic simulation. In particular, $\hat{k}_{jk}$ is the $1-\alpha$ percent quantile of $\left\{\left|t_{jk-PMG}^{(r)}\right|\right\}_{r=1}^{R}$, where $t_{jk-PMG}^{(r)} = \tilde{\beta}_{jk-PMG}^{(r)}/\widehat{se}\left(\hat{\beta}_{PMG}^{(r)}\right) = T^{-1}n^{-1/2}\tilde{\beta}_{jk-PMG}^{(r)}/\hat{\Omega}_{PMG}^{(r)}$, $\tilde{\beta}_{jk-PMG}^{(r)}$ is the jackknife bias-corrected PMG estimate of β using the *r-th* draw of the simulated data generated using the same algorithm as in Subsection S-1.1, and $\hat{\Omega}_{PMG}^{(r)}$ is computed as in (S.4), but using the simulated data.

References

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